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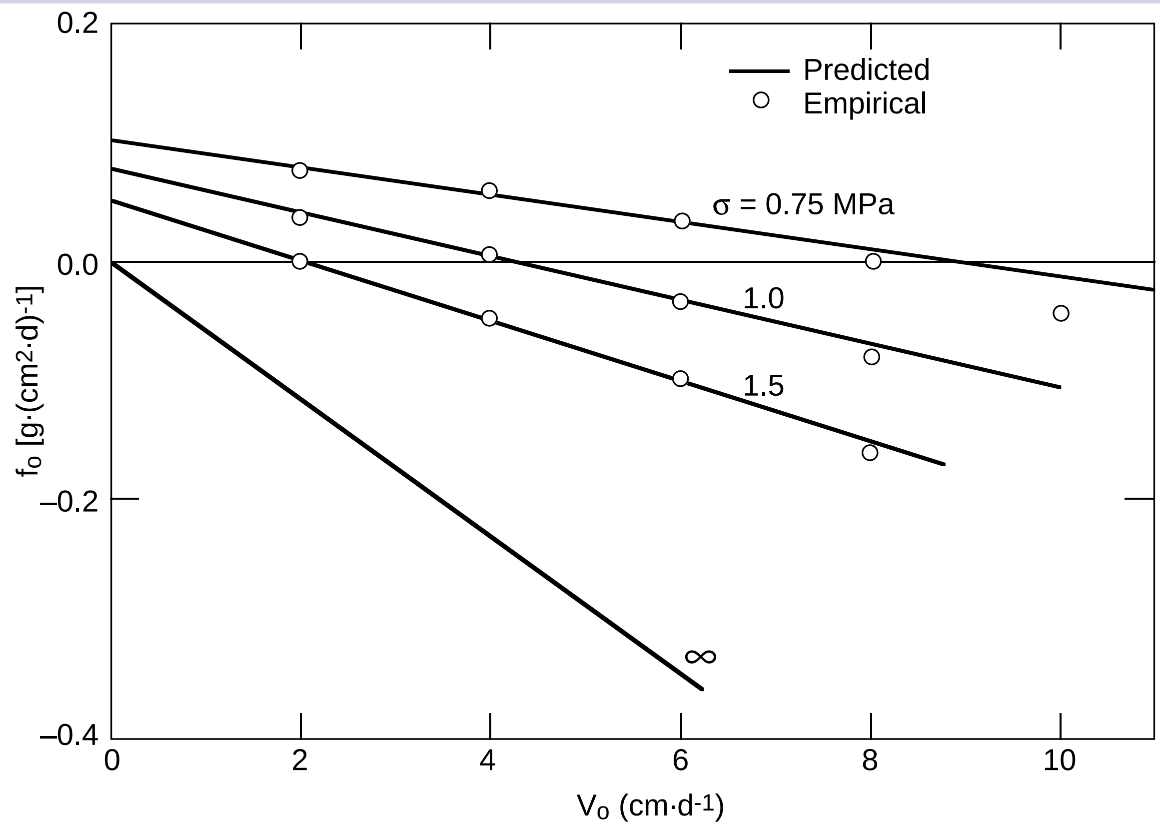
Existence of Traveling Wave Solutions to the Problem of Soil Freezing Described by a Model Called M_1

Yoshisuke Nakano

April 1999



Abstract: The scientific study of soil freezing began in the early 1900s and an accurate mathematical description of the freezing process has been sought for nearly 80 years. Despite numerous publications on the subject, as yet there is no clear consensus on the mathematical model of soil freezing. In this report a mathematical model called M_1 is presented. The existence of traveling wave solutions to the problem is shown. For a given fine-grained soil, such solutions are shown to exhibit three distinct behaviors depending on given thermal and hydraulic conditions. When a frost front (0°C isotherm) advances, water is either attracted to the front or expelled from it. Under certain conditions an ice layer containing hardly any soil particles grows. The report describes how the traveling wave solutions have been used for the empirical verification of M_1.

Cover: Calculated values of f_o *[g/(cm² · d)] vs.* V_o *(cm/d) with* $a_o = 0.75$*°C/cm and* $\delta_o = 1.0$ *cm and* $\sigma = 0.75, 1.0, 1.5$ *MPa, and* ∞.

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Prepared for
OFFICE OF THE CHIEF OF ENGINEERS

PREFACE

This report was prepared by Dr. Yoshisuke Nakano, Chemical Engineer, of the Applied Research Division, U.S. Army Cold Regions Research and Engineering Laboratory. Funding was provided by DA Project AT24-SC-F01, *Physical Processes in Frozen Soil.*

The author thanks Dr. Devinder Sodhi and Dr. Virgil Lunardini for technically reviewing this report.

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ILLUSTRATIONS

Figure

NOMENCLATURE

a_o, a_1	defined by eq 177 and 187
a_2, a_3	defined by eq 188 and 199
a_4	defined by eq 200
A	a small negative number
A_o, A_1	positive numbers used in eq 211
b_i	positive number defined by eq 212–214 where i = 1, 2, 3
B_i	i^{th} constituent of the mixture. Subscripts i = 1, 2, and 3 are used to denote unfrozen water, ice and soil minerals, respectively
c	heat capacity of the mixture defined by eq 8
c_o	defined by eq 30
c_i	heat capacity of the i^{th} constituent
C_s	defined by eq 69
d	unit of time, day
d_i	density of the i^{th} constituent
e_o, e_1	defined by eq 88 and 94
e_2, e_3	defined by eq 119 and 112
$e_4, \hat{e}_4, e_5$	defined by eq 114, 145, and 197
e_6, e_7	defined by eq 198 and 208
e_{1s}, e_{os}	defined by eq 203 and 209
E_1, E_2	defined by eq 121 and 122
E_3, E_4	defined by eq 139 and 140
f	mass flux of water in R_1
f_o	mass flux of water in R_o
f_i	mass flux of the i^{th} constituent relative to that of soil minerals where i = 1, 2
f_s	defined by eq 68
F	defined by eq 220
g_o, g_1	defined by eq 127 and 99
g_2, g_3	defined by eq 100 and 144
h_1, h_2	defined by eq 63 and 64
h_3, h_4	defined by eq 77 and 78
h, h_w	defined by eq 158 and 159
k	thermal conductivity of the mixture
k_o	thermal conductivity in R_o
k_1	thermal conductivity in R_2
K_o	hydraulic conductivity in the unfrozen part of the soil
K_i	empirical function defined by eq 36 where i = 1, 2
K_{20}	defined by eq 214
$\ell_p, \ell(\hat{\alpha}_o)$	defined by eq 179 and 72
L	latent heat of fusion of water, 334 J/g
$L(\alpha_{1e}, \hat{\alpha}_o)$	defined by eq 73
$L(\alpha_{1s}, \alpha_o)$	defined by eq 117
L_c, L^+	defined by eq 126 and 162
L_c^+, L_c^-	defined by eq 135 and 185
L_e, L_p	defined by eq 52 and 172
m_i	positive number defined by eq 216 and 217
$M_1, \hat{M}_1$	names of models
n	boundary in R_o
$\dot{n}$	velocity of $n = dn/dt$
n_i	boundary with i = 0, 1 where n_o denotes the boundary where T = 0 (°C) and n_1 the interface between R_2 and a frozen fringe
P	pressure of water
P_a	applied confining pressure
P_n	$P(n)$
P_0	$P(n_0)$
q	heat flux in the mixture by conduction defined by eq 5
Q	defined by eq 91
r	rate of frost heave
R_o	unfrozen part of the soil
R_1	frozen fringe
R_{10}	part where $0 > T \geq T_\sigma$
R_{11}	part where $T_\sigma > T \geq T_1$
R_2	frozen part of the soil
S_f, S_i	defined by eq 55 and 70
S_m, S_p	defined by eq 53 and 85
S_p^+, S_p^-,	defined by eq 133 and 134
S_{pp}, S_x	defined by eq 174 and 173
s_1	defined by eq 196
s_2	defined by eq 22
s_3	defined by eq 88
t	time
T	temperature of the mixture
T_1	$T(n_1)$
T_a, T_b	temperature at the top and the bottom of a sample
T_c, T_m	defined by eq 131 and 40
T_p, T_s	defined by eq 166 and 68
T_σ, T_x	defined by eq 37 and 97

u_i	velocity of the i^{th} constituent where $i = 1, 2, 3$
V	defined by eq 16
V_0	constant speed of n_0
w_o	defined by eq 44
w_1 , w_2	defined by eq 194 and 195
W	defined by eq 115
W_o , W_1	defined by eq 152 and 192
x	spatial coordinate
y , Y	defined by eq 89
z	defined by eq 17
$\alpha(t)$	trajectory in Figure 3
α_o	absolute value of the temperature gradient at n_o
α_1	absolute value of the limiting temperature gradient as ξ approaches n_1 while ξ is in R_2
α_{1e}, α_{1s}	defined by eq 72 and 73
α_{oc}, α_{1p}	defined by eq 126 and 166
γ	constant, 1.12 (MPa / °C)
δ	thickness of a frozen fringe
δ_o , η	defined by eq 25 and 34
θ_i	volumetric content of the ith constituent
λ_1	rate of supply of mass of the i^{th} constituent per unit volume of the mixture
μ , μ_o	defined by eq 104 and 116
Λ	function defined by eq 31
ν , ν_1	defined by eq 19 and 89
$\hat{\nu}_1$	defined by eq 157
ξ	coordinate defined by eq 10
π_o , π_1	defined by eq 44
ρ_i	bulk density of the i^{th} constituent
ρ_{io}	ρ_i in R_o
σ , σ_o	defined by eq 38 and 24
σ_x , σ_c	defined by eq 129 and 130

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Existence of Traveling Wave Solutions to the Problem of Soil Freezing Described by a Model Called M_1

YOSHISUKE NAKANO

INTRODUCTION

The scientific study of soil freezing and ice segregation began in the early 1900s. By the 1930s researchers (Taber 1930, Beskow 1935) had already found that ice segregation and the resultant frost heave are caused not only by freezing of in-situ water, but also by freezing of water transported toward a freezing front from the unfrozen part of the soil. The understanding gained in the 1930s was largely qualitative. However, the transport of water was already identified as one of major issues in the study of soil freezing. The problem has attracted the attention of many researchers (see Nakano 1991).

The main constituents of saturated, frozen, and fine-grained soils are a solid porous matrix of soil particles and ice, and water in the liquid phase called unfrozen water. The physical properties of all constituents except unfrozen water are well understood. It is generally understood that the transport of water in frozen soils is mainly caused by the movement of unfrozen water and that unfrozen water exists in small spaces surrounded with surfaces of soil particles and ice. Heaving during freezing is not limited to water in soil systems. It occurs with benzene or nitrobenzene in soils (Taber 1930), water in various powder materials including hydrophobic carborundum (Horiguchi 1977), liquid helium in porous glasses (Hiroi et al. 1989), water in hydrophobic silicon-coated glass beads (Sage and Porebska 1993), and water in porous rocks (Miyata et al. 1994).

The dynamic and thermodynamic properties of liquids have been known to be modified by confinement in very small spaces, such as porous media, cell membranes, etc. The problem of confined liquids has attracted the attention of researchers in many disciplines in recent years (Granick 1991). The maximum size of confining space that significantly modifies the property of liquid evidently depends on a kind of liquid and its confining solid. For instance, in the case of thin quartz capillaries with sizes of the order of a micron, the melting point of ice is practically the same as that of bulk ice (Churaev et al. 1993). However, in much smaller capillaries of the order of 50 nm, the melting point changes.

A significant modification may occur in the dynamic behavior of water in fine porous media. It is known (Angell 1983) that the temperature dependence of the self-diffusivity of supercooled water can be described by a critical type of equation with a singular temperature just below the homogeneous nucleation temperature. Recently Teixeira et al. (1997) have found that the self-diffusivity of supercooled water confined in fine porous silica corresponds to that of supercooled water at about 30°C lower temperature. Pagliuca et al. (1987) have shown empirically that the gradients of pressure and temperature are two independent driving forces of water flowing through various noncharged, fine porous, and either hydrophilic or hydrophobic membranes with pore size of the order of 10–500 nm at temperatures above the bulk melting point.

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The specific surface area of fine-grained soils is on the order of 20–200 m^2/g, and unfrozen water is known to exist in the form of thin films. The thickness of such films depends on temperature and pressure, and is estimated on the order of 10–100 nm at the temperatures around – 0.1°C under atmospheric conditions (Ishizaki et al. 1994). Unfrozen water in frozen soils is one special case of a wide class of confined liquids. The key issue underlying the transport of unfrozen water is deemed to be the dynamic collective behavior of water confined to small spaces in frozen soils, which depends on complex solid–liquid interactions.

It is generally accepted that a thin transitional zone, often referred to as the frozen fringe, exists between the 0°C isotherm (frost front) and the growing surface of an ice layer. The unfrozen water content in frozen soils under equilibrium conditions is routinely measured by nuclear magnetic resonance, differential scanning calorimetry, or time domain reflectometry. However, since the unfrozen water content under dynamic conditions is difficult to measure, the phase composition of a frozen fringe is not known. Since the properties of all parts except the frozen fringe are understood, the dynamic behavior of the frozen fringe has been one of the major subjects in the study of soil freezing in recent years. Since the 1960s, many mathematical models (Talamucci 1977, Kay and Perfect 1988) of a frozen fringe have been proposed on the basis of various hypotheses. With the widespread use of computers, the methods of numerical analysis became very popular. However, because of the paucity of basic knowledge and the complex nature of the problem, these numerical studies have not been effective for the critical evaluation of the multiple hypotheses used.

Around 1980, two important semiempirical models of soil freezing were introduced for engineering applications: the segregation potential (SP) model (Konrad and Morgenstern 1981) and the Takashi model (Takashi et al. 1978). Today the SP model is widely used for engineering in Europe and North America, while the Takashi model is the standard of engineering design in Japan. These two semiempirical models share a common approach that the freezing characteristics of a given soil are determined empirically under certain quasi-steady conditions, where a frost front moves with a constant speed. These models also share a common weakness of requiring one or more empirically determined parameters. These are known to depend on not only the properties of a given soil but also a particular quasi-steady condition specified by given thermal and hydraulic fields. The empirical determination of such dependence is elaborate and costly. An accurate mathematical model is needed that provides the functional dependence of parameters on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) describing the properties of a given soil.

As the 1980s were ending, there were many mathematical models of soil freezing (Gilpin 1980, O’Neill and Miller 1985, Fowler 1989, etc.), but they all suffer from the common fault of little or no experimental verification. Efforts were initiated to study the problem analytically and to verify the hypotheses used in the analysis by comparing the property and the behavior of solutions with empirical findings. Adopting such an approach, Nakano (1990) introduced a mathematical model called M_1. This model was shown (Nakano and Takeda 1991, 1994) to be consistent with experimental data on the growth condition of an ice layer without overburden load (Takeda and Nakano 1990) and under load (Takeda and Nakano 1993). The growth process of final ice lenses was accurately described by M_1 (Nakano 1992, Nakano and Takeda 1993). Nakano (1994b) has shown that the functional dependence of SP on thermal and hydraulic conditions predicted by M_1 is consistent with empirical findings that were used to build the SP model.

According to the Takashi model the freezing characteristics of a given soil are described by two empirical formulas that specify the dependence of the frost heave ratio and the water intake ratio on given thermal and hydraulic conditions. Two theoretical equations

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corresponding to Takashi's formulas are derived by using the analytical solution of quasi-steady problems (Nakano 1994a, Nakano and Primicerio 1995). Comparing the theoretical formulas with the empirical ones for Kanto loam, Nakano (1996) has shown that M_1 is compatible with the Takashi model. Studying the property of a frozen fringe described by the Gilpin model (Gilpin 1980), Nakano (1997) has shown that the Gilpin model is essentially one special case of M_1 and that it is too restrictive to accurately describe the behavior of two kinds of porous media studied. Assuming linear temperature profiles in both frozen and unfrozen parts and neglecting the effect of changing composition in the frozen fringe, Talamucci has solved the first (Talamucci 1998a) and second (Talamucci 1998b) boundary value problems of unsteady soil freezing based on M_1.

In this work the problem of soil freezing is studied by using M_1. We will show that traveling wave solutions to the problem exist and describe how these solutions have been used for the empirical verification of M_1.

BALANCE EQUATIONS OF MASS AND HEAT

We will consider the one-directional freezing of soils. Let the freezing process advance from the top down and the coordinate x be positive upward with its origin fixed at some point in the unfrozen part of the soil. We will treat the soil as a mixture of water in the liquid phase B_1, ice B_2 , and soil minerals B_3. The bulk density of B_i is denoted by $\rho_i\,(x,t)$. If d_i is the density of the ith constituent, then the volumetric content $\theta_i\,(x,t)$ of the ith constituent is given as

$$\theta_i = \rho_i / d_i. \tag{1}$$

It is clear that the sum of θ_i should be unity, namely:

$$\theta_1 + \theta_2 + \theta_3 = 1. \tag{2}$$

We will assume that the density of each constituent remains constant.

We will assume that the unfrozen part of the soil is kept saturated with water at all times by using an appropriate water supply device. The balance of mass for the i^{th} constituent is given as (Nakano 1990)

$$\frac{\partial}{\partial t}\rho_i = -\frac{\partial}{\partial x}\left(\rho_i u_i\right) + \lambda_i, \quad i = 1,2,3 \tag{3}$$

where $u_i\,(x,t)$ is the velocity of the i^{th} constituent, and $\lambda_i(x, t)$ the time rate of supply of mass of the ith constituent per unit volume of the mixture. The summation convention on index i is not in force here, so that $(\rho_i u_i)$ represents only one term. Since none of the constituents is involved in a chemical reaction, we have

$$\lambda_1 + \lambda_2 = 0 \qquad \text{and} \qquad \lambda_3 = 0. \tag{4}$$

We will assume that the constituents are locally in thermal equilibrium with each other and that the heat capacity c_i of the ith constituent and the latent heat of fusion of water L do not depend on the temperature T. If k is the thermal conductivity of the mixture, the conductive heat flux $q\,(x,t)$ in the mixture is assumed to be given as

$$q = -k\frac{\partial T}{\partial x}. \tag{5}$$

Using eq 5, we will obtain the balance equation of heat for the mixture (Nakano 1990) given as

$$\frac{\partial}{\partial x} q = L\left(\lambda_2 + z\right) \tag{6}$$

where $z\,(x,t)$ is defined as

$$Lz = -c\,\frac{\partial T}{\partial t} + (c_1 - c_2)T\,\lambda_2 - \sum_i \rho_i u_i c_i \frac{\partial T}{\partial x} \tag{7}$$

$$c = c_1\rho_1 + c_2\rho_2 + c_3\rho_3 . \tag{8}$$

We will consider a special case in which a frost front $x = n_0\,(t)$ moves with a constant speed, namely

$$-\frac{d}{dt} n_o(t) = -\dot{n}_o = V_o \geq 0. \tag{9}$$

Hereafter we will exclude the case of negative V_o where melting occurs. We will introduce a new independent variable ξ defined as

$$\xi = x - \dot{n}_o t - n_o(0). \tag{10}$$

For the sake of convenience we will define new dependent variables $f_1\,(\xi)$ and $f_2\,(\xi)$ as

$$f_1 = \rho_1(u_1 - u_3) \tag{11}$$

$$f_2 = \rho_2(u_2 - u_3). \tag{12}$$

Therefore, f_i (i = 1,2) is the mass flux of either B_1 or B_2 relative to the mass flux of soil particles. Using eq 10, 11 and 12, we reduce eq 3 to

$$(\rho_1 V)' = -f_1' - \lambda_2 \tag{13}$$

$$\left(\rho_2 V\right)' = -f_2' + \lambda_2 \tag{14}$$

$$\left(\rho_3 V\right)' = 0 \tag{15}$$

where primes denote differentiation with respect to ξ and $V\,(\xi)$ is defined as

$$V = u_3 - \dot{n}_0. \tag{16}$$

Similarly we will reduce eq 6 and 7 to

$$q' = -\,(kT')' = L\left(\lambda_2 + z\right) \tag{17}$$

$$Lz = -\left(c_1 f_1 + c_2 f_2 + cV\right)T' + \left(c_1 - c_2\right)\lambda_2 T. \tag{18}$$

QUASI-STEADY PROBLEM

A freezing soil may be considered to consist of three parts: the unfrozen part R_o, the frozen fringe R_1 and the frozen part R_2, as shown in Figure 1. We will also make seven assumptions:

1. The dry density of R_o remains constant,
2. The composition is continuous at n_o,
3. The pressure P of water at n remains constant at P_n ,
4. f_2 vanishes in R_1 and R_2 unless ρ_3 vanishes,
5. The flux f_1 is negligibly small in R_2,
6. Sensible heat terms are negligible in comparison with latent heat terms,
7. ρ_1 is given in R_1 and R_2 as

$$\rho_1 = \rho_3 \nu(T). \tag{19}$$

The bulk density ρ_1 under equilibrium conditions is known to be given by eq 19 where $\nu(T)$ is an empirically determined and increasing function of T. Hence, the assumption 7 implies that ρ_1 under dynamic conditions is also given by the same form as eq 19. We will assume that $\nu(T)$ has a continuous first derivative.

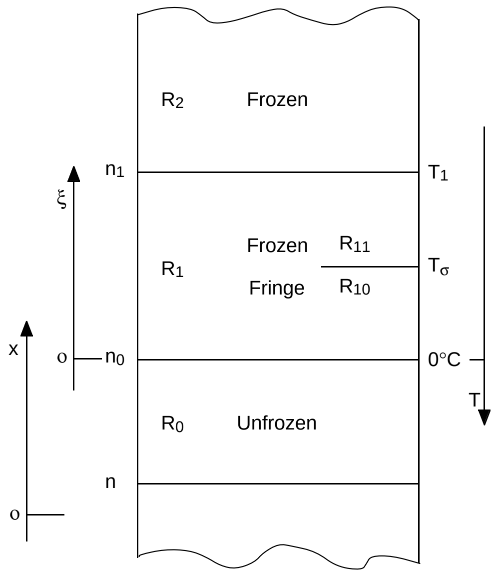


Figure 1. Quasi-steady freezing of soil.

We will seek a traveling wave solution to the problem in which the boundaries $n(t)$, $n_o(t)$ and $n_1(t)$ move with the same constant speed V_o, namely

$$V_o = -\dot{n} = -\dot{n}_o = -\dot{n}_1. \tag{20}$$

From a physical point of view, maintaining a constant pressure P_n is difficult at the moving boundary $n(t)$. However, a solution obtained under such an idealized condition is quite useful for applications (Nakano and Primicerio 1995). If such a solution exists, it must satisfy eq 13– 15, and eq 17.

From eq 13, 14, and 15, we find that the flux of water $f_1(\xi)$ is given in R_1 (Nakano 1994a) as

$$f_1 = f_o + s_2\left(\rho_{10} - \rho_{30}\nu\right)V_o - d_2(V - V_o), \quad 0 < \xi < \delta \tag{21}$$

where ρ_{10} and ρ_{30} are the constant bulk densities of B_1 and B_3 in R_o, respectively, and $\delta = n_1 - n_o$, and f_o is the constant flux of water in R_o. s_2 is defined as

$$s_2 = 1 - d_1^{-1} d_2. \tag{22}$$

Neglecting the gravitational effects and using Darcy's law, the flux of water f_o in R_o is given as

$$f_o = K_o \sigma_o \delta_o^{-1} \tag{23}$$

where K_o is the hydraulic conductivity of R_o. σ_o and δ_o are defined as

$$\sigma_o = P_n - P_o, \quad P_o = P(0) \tag{24}$$

$$\delta_o = n_o - n. \tag{25}$$

The boundary n_1 is a free boundary. The composition may be discontinuous at n_1 and the limiting value ρ_3 (δ+) of ρ_3 as ξ approaches δ, while ξ is in R_2 is given (Nakano 1994a) as

$$\rho_3(\delta+) = \rho_{30} V_o [r(\delta+) + V_o]^{-1} \tag{26}$$

where r (δ+) is the rate of heave at δ+ given as

$$r(\delta+) = d_2^{-1} f_o + d_2^{-1} s_2 [\rho_{10} - \rho_{30} \nu\{T(\delta)\}] V_o. \tag{27}$$

The heat flux is discontinuous at n_1 and the jump condition is given as

$$q(\delta+) = q(\delta-) + f_1(\delta-)[L + (c_1 - c_2)T_1] \tag{28}$$

where $q(\delta-)$ and $f_1(\delta-)$ are limiting values of q and f_1, respectively, as ξ approaches δ, while ξ is in R_1 and T_1 is T (δ).

We will reduce eq 17 and eq 18 to a simpler form. Using eq 13, 14, and 15, we obtain

$$cV = c_o V_o - (c_1 - c_2)\Lambda + c_1(f_o - f_1) \tag{29}$$

where

$$c_o = c_1 \rho_{10} + c_3 \rho_{30} \tag{30}$$

$$\Lambda(\xi) = \int_0^{\xi} \lambda_2 d\xi, \qquad \xi \geq 0. \tag{31}$$

Using eq 29, neglecting a sensible heat term, and integrating eq 17, we obtain

$$-k\,T' = k_o \alpha_o + L\Lambda, \quad 0 < \xi < \delta \tag{32}$$

where k_o and k are the thermal conductivities of R_o and R_1, respectively, and $\alpha_o \geq 0$ is the absolute value of the temperature gradient at ξ = 0. Using eq 32 and neglecting sensible heat terms, we will reduce eq 28 to

$$k_1 \alpha_1 - k_o \alpha_o = \left[f_1(\delta-) + \Lambda(\delta-)\right] L \tag{33}$$

where k_1 is the thermal conductivity of R_2 and α_1 is $-T'$ (δ+).

Using the principle of mass and heat conservation, we have derived equations that must be satisfied by a traveling wave solution of soil freezing. Clearly these equations are not sufficient to solve the problem. We need a model of a frozen fringe that specifies f_1 (ξ) and Λ (ξ).

MODEL STUDY

A model of a frozen fringe called M_1 was introduced by Nakano (1990) to explain empirical findings on the growth condition of an ice layer in freezing soils. The model has been modified as its empirical evaluation has progressed (Takeda and Nakano 1990, Nakano and Takeda 1991, Takeda and Nakano 1993, Nakano and Takeda 1994). The latest version assumes the validity of equations in R_1 given as

$$k = \text{constant}, \qquad k / k_o = \eta \geq 1 \tag{34}$$

$$\rho_3 = \rho_{30}\,, \quad \rho_1 = \nu(T)\rho_{30} \le \rho_{10}\,, \quad \rho_1(0+) = \rho_{10} \tag{35}$$

$$f \equiv f_1 = -K_1 P' - K_2 T'\,, \quad K_1(0+) = K_0 \tag{36}$$

$$K_2(T) \,/\, K_1(T) = \gamma \quad \text{for} \quad 0 > T \ge T_\sigma = -\sigma \,/\, \gamma \tag{37}$$

$$P(\delta-) = P_a = \sigma + P_n \;, \quad \sigma \ge 0 \tag{38}$$

$$P'(\delta-) \ge 0, \quad V_o \ge 0 \quad \text{and} \quad V_o P'(\sigma-) = 0 \tag{39}$$

$$K_2(T) = 0, \quad K_2(T) \,/\, K_1(T) = 0 \quad \text{for} \quad T \le T_m < T_\sigma \tag{40}$$

where γ is a constant (1.12 MPa / °C), $P(\xi)$ is the pressure of water, P_a is the applied confining pressure (uniaxial stress), σ is the effective confining pressure, and K_i (i = 1,2) is the transport property of a given soil that generally depends on the temperature and the composition of the soil. Since ρ_3 is a constant, we will assume that K_i is an increasing function of T alone. This assumption implies the homogeneity of soils in a microscopic scale that corresponds to the thickness of the frozen fringe, which is clearly an approximation. We will assume that $K_1(T)$ has a continuous first derivative. Because of eq 37, $K_2(T)$ may be discontinuous at $T = T_\sigma$. We will assume that the first derivative of K_2 is continuous except at $T = T_\sigma$. It is known that the mobility of unfrozen water tends to diminish as T decreases. We will assume that there exists a negative number $T_m < T_\sigma$ such that eq 40 holds true and that $K_2(T) > 0$ and $K_2(T)/K_1(T) > 0$ for $T > T_m$. According to M_1, f is given by eq 36 in R_1 while Darcy's law holds true in R_o. Hence, f and P are continuous but P' may be discontinuous at n_o.

The M_1 model is a generalization of somewhat simpler but more restrictive models, $\hat{M}_1$ (Derjaguin and Churaev 1978, Ratkje et al. 1982, Horiguchi 1987), in which the ratio K_2/K_1 is equal to γ regardless of T. In $\hat{M}_1$ the coupling mechanism for mass and heat transport is based on irreversible thermodynamics in which local equilibrium is assumed under a temperature gradient (Ratkje and Hafskjold 1996). In M_1 local equilibrium holds in the part R_{10} where $T_\sigma \le T < 0$, but not in the part R_{11} where $T(\delta) < T < T_\sigma$ (Fig. 1). This generalization is needed because $\hat{M}_1$ is too restrictive to accurately describe the behavior of porous media (Nakano 1997). Equation 39, often referred to as the Signorini-type free boundary condition (Friedmann and Jiang 1984), is needed for the uniqueness proof of solutions when V_o is positive. It is not certain that such a condition holds true because of the paucity of experimental data. In addition to the above equations, we will assume that the thermal conductivities k_o and k_1 are given constants for the sake of simplicity.

When eq 35 holds true, u_3 vanishes and eq 21 is reduced to

$$f(\xi) = f_o + s_2(\rho_{10} - \rho_{30}\nu)V_o, \quad 0 < \xi < \delta. \tag{41}$$

The $\Lambda(\xi)$ is given as

$$\Lambda(\xi) = d_1^{-1} d_2(\rho_{10} - \rho_{30}\nu)V_o. \tag{42}$$

According to M_1 the properties of a given soil are described by three empirically determined functions of T : K_1, K_2 and v that are assumed to be functions of T alone for $T < 0$°C. The hydraulic field is specified by P_n, δ_o and P_a while the thermal field is specified by α_o and α_1. Our problem is to find constants $V_o \ge 0$, $\delta \ge 0$ and functions $f(\xi)$, $T(\xi) \le 0$, $P(\xi)$ so that the following equations (P1 through P7) are satisfied:

From eq 41 we have

$$f(\xi) = f_o + s_2\left[\rho_{10} - \rho_{30}\nu\{T(\xi)\}\right]V_o \quad , \quad 0 < \xi < \delta. \tag{P1}$$

From eq 36 we have

$$f(\xi) = -K_1\{T(\xi)\}P'(\xi) - K_2\{T(\xi)\}T'(\xi) \quad , \qquad 0 < \xi < \delta. \tag{P2}$$

From eq 32 and 49 we have

$$kT'(\xi) = -k_o\alpha_o - d_1^{-1}d_2\left[\rho_{10} - \rho_{30}\nu\{T(\xi)\}\right]LV_o , \quad 0 < \xi < \delta. \tag{P3}$$

From eq 33 and 42 we have

$$k_1\alpha_1 - k_o\alpha_o = Lf_o + \left[\rho_{10} - \rho_{30}\nu\{T(\delta)\}\right]LV_o. \tag{P4}$$

Boundary conditions are given as

$$T(0) = 0 \tag{P5}$$

$$P(\delta -) = P_a \tag{P6}$$

$$P'(\delta -) \geq 0, \qquad V_o \geq 0 \quad \text{and} \quad V_o P'(\delta -) = 0. \tag{P7}$$

We will rewrite eq P3 as

$$\eta T' = -\pi_1 + \pi_o \nu \tag{43}$$

where π_o and π_1 are given as

$$\pi_o = d_1^{-1}d_2\, k_o^{-1}\rho_{30}LV_o , \quad \pi_1 = \alpha_o + \pi_o w_o , \quad w_o = \rho_{10} / \rho_{30}. \tag{44}$$

Since $\nu(T) < w_o$ for $T < 0°\mathrm{C}$ and $\alpha_o \geq 0$, $T'(\xi)$ is strictly negative. Hence the function $T(\xi)$ is invertible for $\delta \geq \xi > 0$. Integrating eq 43 by using eq P5, we obtain

$$T(\xi) = -(\pi_1 / \eta)\xi - (\pi_o / \pi_1)\int_0^T \nu\left[1 - (\pi_o / \pi_1)\nu\right]^{-1} dT. \tag{45}$$

Integrating eq P2, we obtain

$$P[\xi(T)] - P_n + (\delta_o / K_o)f_o = -\int_0^T (K_2 / K_1)\,dT - \int_0^T f[\xi(T)](K_1 T')^{-1}\,dT. \tag{46}$$

Setting $T = T_1$ in (46) and using eq P6, we obtain

$$\sigma + (\delta_o / K_o)f_o = -\int_{T_1}^0 (K_2 / K_1)\,dT + \int_{T_1}^0 f[\xi(T)](K_1 T')^{-1}\,dT. \tag{47}$$

Equation 47 provides the functional dependence of T_1 on α_o and α_1 that specifies a given thermal condition as well as on δ_o and σ that specifies a given hydraulic condition in terms of functions and parameters, such as K_1, K_2, and ν, etc., describing the properties of a given soil.

GROWTH OF ICE LAYERS

We will seek solutions in which $V_o = 0$. In this case eq P1 through P4 are reduced to

$$f(\xi) = f_o \quad , \quad 0 < \xi < \delta \tag{48}$$

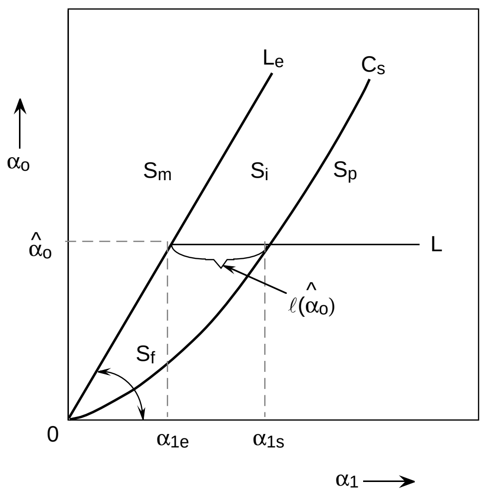
$$f(\xi) = -K_1\{T(\xi)\}P'(\xi) + K_2\{T(\xi)\}(\alpha_o / \eta) \quad , \qquad 0 < \xi < \delta \tag{49}$$

$$T'(\xi) = -(\alpha_o / \eta) \tag{50}$$

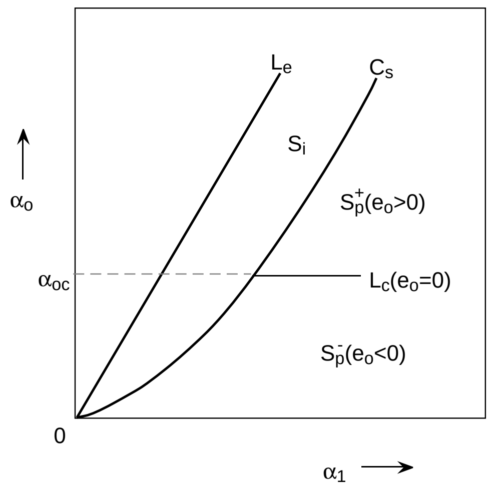
$$k_1\alpha_1 - k_o\alpha_o = Lf_o . \tag{51}$$

The left-hand side of eq 51 is the rate of heat removal from the frozen fringe that must be positive during soil freezing. Hence, $f_o > 0$. We will consider a quadrant $S = [(\alpha_1, \alpha_o)\colon \alpha_1 \geq 0, \alpha_o \geq 0]$, where we draw a straight line L_e starting from the origin (Fig. 2a) defined as

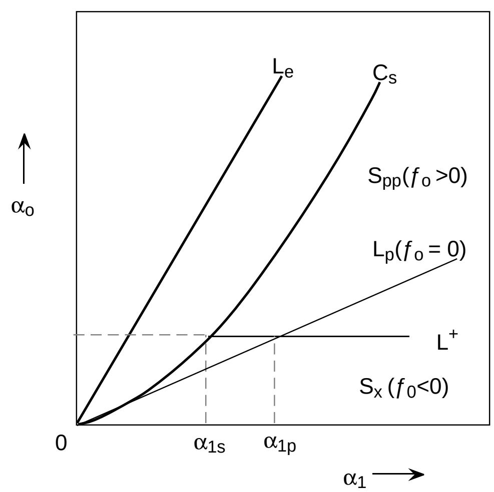
$$L_e = \{(\alpha_1, \alpha_o)\colon \alpha_o = (k_1 / k_o)\alpha_1\}. \tag{52}$$



a. Stable growth region of an ice layer.



b. Region of frost penetration when $\sigma < \sigma_c$.



c. Region of frost penetration when $\sigma \geq \sigma_c$

Figure 2. Temperature gradients α_1 *and* α_o.

It follows from eq 51 that f_o vanishes on L_e.The line L_e divides S into two regions, and we will denote one of them by S_m defined as

$$S_m = \{(\alpha_1, \alpha_o) : \alpha_o > (k_1 / k_o)\alpha_1\}. \tag{53}$$

Therefore, melting takes place in S_m. We will exclude S_m from our discussion hereafter. For a special case where $\alpha_o = 0$ and $f_o \geq 0$, from eq 49 we find

$$f_o = -K_1(T_1)P'(\delta-). \tag{54}$$

It follows from eq 54, P7 and 51 that α_1 and f_o also vanish. Now we will seek solutions with $V_o = 0$ in the region S_f defined as

$$S_f = S - (S_m + L_e) = \{(\alpha_1, \alpha_o) : 0 < \alpha_o < (k_1 / k_o)\alpha_1\}. \tag{55}$$

Suppose that such solutions exist in S_f. Then, eq 26 and 27 are reduced to

$$\rho_3(\delta+) = 0 \tag{56}$$

$$r(\delta+) = d_2^{-1} f_o > 0. \tag{57}$$

It follows from eq 56 and 57 that an ice layer grows in the solutions

When $V_o = 0$ in S_f , eq 47 is reduced to

$$\sigma + \left(\delta_o / K_o\right) f_o = \int_{T_1}^{0} (K_2 / K_1) dT - (\eta / \alpha_o) f_o \int_{T_1}^{0} (K_1)^{-1} dT. \tag{58}$$

We will write eq 58 as

$$\left[\left(\delta_o / K_o\right) + (\eta / \alpha_o) \int_{T_1}^{0} (K_1)^{-1} dT\right] f_o = \int_{T_1}^{0} (K_2 / K_1) - \sigma. \tag{59}$$

Suppose that $T_1 \geq T_\sigma$, using eq 37, we obtain

$$\int_{T_1}^{0} (K_2 / K_1) dT - \sigma = \gamma(T_\sigma - T_1) \leq 0, \quad \text{if } T_1 \geq T_\sigma. \tag{60}$$

It follows from eq 59 and eq 60 that $f_o \leq 0$. Therefore, if there exists T_1 such that $f_o > 0$ and eq 58 holds true, then T_1 must be less than T_σ. Our next aim is to find points (α_1, α_o) in S_f with $T_1 < T_\sigma$ and $V_o = 0$. Hereafter, we will assume that $\sigma \geq 0$ and $\delta_o > 0$ are given constants.

By eq P7 P' (δ –) is nonnegative if $V_o = 0$. We will begin our study with a special case in which P' (δ –) vanishes.

Proposition 1

If P' (δ –) vanishes for a given α_o, then there exists a unique T_1 such that $T_m < T_1 < T_\sigma$ and that eq 58 holds true.

Proof

When P' (δ –) vanishes, then f_o is given as

$$f_o = K_2(T_1)(\alpha_o / \eta). \tag{61}$$

Using eq 61, we will write eq 58 as

$$h_1(T_1) = h_2(T_1) \tag{62}$$

where h_1 and h_2 are defined as

$$h_1(T_1) = \sigma + (\delta_o / K_o)K_2(T_1)(\alpha_o / \eta) \tag{63}$$

$$h_2(T_1) = \int_{T_1}^{0} [K_2(s) - K_2(T_1)][1 / K_1(s)]ds. \tag{64}$$

Since $f_o > 0$, then $T_1 < T_\sigma$. Using eq 37, we will reduce eq 64 to

$$h_2(T_1) = \sigma + \int_{T_1}^{T_\sigma} \left\{K_2(s) / K_1(s)\right\} ds - K_2(T_1)\int_{T_1}^{0} \left[1 / K_1(s)\right] ds. \tag{65}$$

Differentiating $h_1(T_1)$ and $h_2(T_1)$ with respect to T_1, we obtain

$$\dot{h}_1(T_1) = (\delta_o / K_o)\dot{K}_2(T_1)(\alpha_o / \eta) \tag{66}$$

$$\dot{h}_2(T_1) = -\dot{K}_2(T_1)\int_{T_1}^{0} \left[1 / K_1(s)\right] ds \tag{67}$$

where a dot denotes differentiation with respect to T_1. Therefore, from eq 63 and 65, $h_1(T_m) < h_2(T_m)$ and $h_1(T_\sigma–) > h_2(T_\sigma–)$. From eq 66 and 67 we find that $\dot{h}_1(T_1) > 0$ and $\dot{h}_2(T_1) < 0$ for $T_m < T_1 < T_\sigma$. Also $h_1(T_1)$ and $h_2(T_1)$ are continuous for $T_1 < T_\sigma$. Therefore, there exists a unique T_1 such that $T_1 < T_\sigma$ and that eq 44 holds true. ❑

We will denote the unique T_1 of Prop. 1 for a given α_o by $T_s(\alpha_o)$. It is easy to see that the function $T_s(\alpha_o)$ is continuous for $T_1 < T_\sigma$. If we denote f_o by f_s, when $T_1 = T_s$, then f_s is given as

$$f_s = K_2(T_s)(\alpha_o / \eta). \tag{68}$$

Substituting f_o in eq 51 with f_s, we will define a curve C_s in S_f (Fig. 2a) as

$$C_s = \{(\alpha_1, \alpha_o):\alpha_o = k_1[k_o + \eta^{-1}LK_2\{T_s(\alpha_o)\}]^{-1}\alpha_1\}. \tag{69}$$

We also define the region S_i bounded by L_e and C_s as

$$S_i = \left\{(\alpha_1, \alpha_o):(k_1 / k_o)\alpha_1 > \alpha_o > k_1\left[k_o + \eta^{-1}LK_2\{T_s(\alpha_o)\}\right]^{-1}\alpha_1\right\}. \tag{70}$$

From eq 62, T_s depends on α_o, δ_o, and σ for a given soil. We will show the nature of such dependence below.

Proposition 2

The solution T_s of eq 62 is a decreasing function of α_o, δ_o, $\alpha_o\delta_o$ and σ.

Proof

Differentiating eq 62 with respect to α_o, we obtain

$$\dot{K}_2(T_s)\frac{\partial T_s}{\partial \alpha_o}\left\{\delta_o(\eta K_o)^{-1}\alpha_o + \int_{T_s}^{0}\left[1 / K_1(s)\right]ds\right\} = -\delta_o(\eta K_o)^{-1}K_2(T_s). \tag{71}$$

It follows from eq 71 that T_s is a decreasing function of α_o. Similarly it is easy to find that T_s is a decreasing function of δ_o , $\alpha_o\delta_o$ and σ. ❑

Next we will study the region S_i. For a given $\hat{\alpha}_o$, we will consider a segment $\ell(\hat{\alpha}_o)$ of a straight line $L(\alpha_{ie},\hat{\alpha}_o)$ (Fig. 2a) defined as

$$\ell(\hat{\alpha}_o) = \{(\alpha_1,\alpha_o) : \alpha_o = \hat{\alpha}_o \text{ and } \alpha_{1e} < \alpha_1 < \alpha_{1s}\} \tag{72}$$

$$L(\alpha_{1e},\hat{\alpha}_o) = \{(\alpha_1,\alpha_o) : \alpha_o = \hat{\alpha}_o, \alpha_1 > \alpha_{1e}\} \tag{73}$$

where $(\alpha_{1e},\hat{\alpha}_o) \in L_e$ and $(\alpha_{1s},\hat{\alpha}_o) \in C_s$. The flux $f_o\,(\alpha_1)$ on $\ell(\hat{\alpha}_o)$ is given by eq 51 as

$$f_o(\alpha_1) = (k_1\alpha_1 - k_o\hat{\alpha}_o)\,/\,L. \tag{74}$$

The flux depends linearly on α_1, $f_o(\alpha_1) > 0$ on $\ell(\hat{\alpha}_o)$, $f_o(\alpha_{1e}) = 0$ and $f_o(\alpha_{1s}) = f_s$ where f_s is given as

$$f_s = K_2\{T_s(\hat{\alpha}_o)\}(\hat{\alpha}_o\,/\,\eta). \tag{75}$$

The $P'\,(\delta -)$ vanishes at the point $(\alpha_{1s},\hat{\alpha}_o)$. Suppose that $P'\,(\delta -)$ vanishes at some point on $\ell(\hat{\alpha}_o)$, then there is no solution of eq 58 at that point by Prop. 1. We will seek solutions on $\ell(\hat{\alpha}_o)$ under the condition of $P'\,(\delta -) > 0$.

Proposition 3

For a given $\hat{\alpha}_o$, there exists a unique T_1 on $\ell(\hat{\alpha}_o)$ such that $T_s < T_1 < T_\sigma$ and eq 58 holds true if $P'\,(\delta -) > 0$.

Proof

For a given point $(\alpha_1,\hat{\alpha}_o)$ on $\ell(\hat{\alpha}_o)$ we will write eq 58 as

$$h_3(\alpha_1) = h_4(T_1,\alpha_1) \tag{76}$$

where h_3 and h_4 are defined as

$$h_3(\alpha_1) = \sigma + \left(\delta_o\,/\,K_o\right)f_o(\alpha_1) \tag{77}$$

$$h_4(T_1,\,\alpha_1) = \int_{T_1}^{0}[K_2(s)\,/\,K_1(s)]ds - (\eta\,/\,\hat{\alpha}_o)f_o(\alpha_1)\int_{T_1}^{0}[1\,/\,K_1(s)]ds. \tag{78}$$

Differentiating $h_4\,(T_1,\,\alpha_1)$ with respect to T_1, we obtain

$$\dot{h}_4(T_1,\,\alpha_1) = -(\eta\,/\,\hat{\alpha}_o)P'(\delta -). \tag{79}$$

Since $P'\,(\delta -) > 0$, $h_4(T_1,\,\alpha_1)$ is a decreasing function of T_1. When T_1 approaches $T_s(\hat{\alpha}_o)$, we will evaluate h_4. Since $T_s(\hat{\alpha}_o)$ is the solution of eq 62, from eq 63 and 64 we obtain

$$\sigma + (\delta_o\,/\,K_o)f_s = \int_{T_s}^{0}[K_2(s) - K_2(T_s)][1\,/\,K_1(s)]ds. \tag{80}$$

Using eq 80, from eq 78 we obtain

$$h_4(T_s,\alpha_1) = \sigma + (\delta_o\,/\,K_o)f_s + (\eta\,/\,\hat{\alpha}_o)K_1(T_s)P'(\delta -)\int_{T_s}^{0}[1\,/\,K_1(s)]\,ds. \tag{81}$$

It follows from eq 80 and 81 that $h_4(T_s,\,\alpha_1) > h_3(\alpha_1)$. Also when T_1 approaches T_σ, we have

$$h_4(T_\sigma^-, \alpha_1) = \sigma - (\eta / \hat{\alpha}_o) f_o \int_{T_\sigma}^{0} [1 / K_1(s)] ds < h_3(\alpha_1)]. \tag{82}$$

Therefore, there exists a unique T_1 such that $T_s < T_1 < T_\sigma$ and eq 58 holds true. ❑

Differentiating eq 76 with respect to α_1, we obtain

$$(L / k_1) P'(\delta-) \frac{\partial T_1}{\partial \alpha_1} = -\alpha_o \delta_o (K_o \eta)^{-1} - \int_{T_1}^{0} [1 / K_1(s)] ds. \tag{83}$$

From eq 83 we find on $\ell(\hat{\alpha}_o)$

$$\frac{\partial T_1}{\partial \alpha_1} < 0, \quad \text{and} \quad \frac{\partial T_1}{\partial \alpha_1} \to -\infty \quad \text{as} \ \alpha_1 \to \alpha_{1s}. \tag{84}$$

As α_1 increases from α_{1e} to α_{1s} on $\ell(\hat{\alpha}_o)$, the flux f_o increases from zero to f_s, while T_1 decreases from T_σ to T_s. In Proposition 3 $\hat{\alpha}_o$ is an arbitrary positive number. Hence, we may conclude that an ice layer grows in the region S_i and on C_s. Below we will study the region S_p defined as

$$S_p = \left\{ (\alpha_1, \alpha_o) : k_1 \left[k_o + \eta^{-1} L K_2 \{ T_s(\alpha_o) \} \right]^{-1} \alpha_1 > \alpha_o > 0 \right\}. \tag{85}$$

FROST PENETRATION

We will seek solutions with a positive V_o in S_p. If such solutions exist, by eq P7 $P'(\delta -)$ vanishes and eq P2 is reduced to

$$f(\delta-) = -K_2(T_1) T'(\delta-). \tag{86}$$

Using eq P1, 43 and 86, and neglecting sensible heat terms, we obtain

$$e_o \rho_{10} \left(1 - \nu_1 w_o^{-1}\right) V_o = Y - f_o \tag{87}$$

where

$$e_o = s_2 (1 - \eta^{-1} s_3 y), \quad s_3 = k_o^{-1} L / (d_1 d_2^{-1} - 1) \tag{88}$$

$$\nu_1 = \nu(T_1), \quad Y = y \alpha_o / \eta, \quad y = K_2(T_1). \tag{89}$$

We will write eq P4 as

$$\rho_{10} \left(1 - \nu_1 w_o^{-1}\right) V_o = Q - f_o \tag{90}$$

where

$$Q = (k_1 \alpha_1 - k_o \alpha_o) / L. \tag{91}$$

It should be noted that e_o and Y are functions of T_1 because y is a function of T_1.

Using eq 87 and 90, we will express V_o and f_o in terms of Y and Q as

$$e_1 \rho_{10} (1 - \nu_1 w_o^{-1}) V_o = Q - Y \tag{92}$$

$$e_1 f_o = Y - e_o Q \tag{93}$$

where e_1 is a positive function of T_1 defined as

$$e_1 = 1 - e_o = d_1^{-1} d_2 \left[1 + (k_o \eta)^{-1} L y \right]. \tag{94}$$

Now the problem of finding a solution with positive V_o is reduced to that of finding $T_1 < 0$ that satisfies eq 47, 92 and 93.

It follows from eq 92 and 93 that there are two possible types of solutions satisfying one of the following conditions given as

$$e_o \le 0 \text{ or } e_o > 0 \text{ and } Y \ge e_o Q, \quad \text{then } f_o \ge 0 \tag{95}$$

$$e_o > 0 \text{ and } Y < e_o Q, \quad \text{then } f_o < 0. \tag{96}$$

Since s_3 is a positive number and $K_2(T)$ is an increasing and continuous function if $\sigma = 0$, we will define $T_x < 0$ as

$$K_2(T_x) = \eta / s_3, \ \sigma = 0. \tag{97}$$

Using eq 92 and 93, we will write eq 47 as

$$g_1(T_1) = g_2(T_1) \tag{98}$$

where g_1 and g_2 are defined as

$$g_1(T_1) = \sigma + (\delta_o / K_o) f_o(T_1) \tag{99}$$

$$g_2(T_1) = \int_{T_1}^{0} \frac{K_2(s)}{K_1(s)} ds + \int_{T_1}^{0} \frac{f(s, T_1)}{K_1(s) T'(s, T_1)} ds \tag{100}$$

and T' and f are given as

$$T'(s, T_1) = -\eta^{-1} \left[\alpha_o + s_2 s_3 \mu(s, T_1) \{Q - Y(T_1)\} / e_1(T_1) \right] \tag{101}$$

$$f(s, T_1) = f_o(T_1) + s_2 \mu(s, T_1) \{Q - Y(T_1)\} / e_1(T_1) \tag{102}$$

$$f_o(T_1) = \left[Y(T_1) - e_o(T_1) Q \right] / e_1(T_1) \tag{103}$$

$$\mu(s, T_1) = \left[w_o - \nu(s) \right] / \left[w_o - \nu(T_1) \right]. \tag{104}$$

It is noted that g_1 and g_2 may be discontinuous at $T_1 = T_\sigma$ due to the singularity of $K_2(T_1)$. Below we will study the properties of $g_1(T_1)$ and $g_2(T_1)$ that will be used later for existence proofs.

Proposition 4

For given α_o and α_1, $g_1(T_1)$ and $g_2(T_1)$ have the following properties:

$$g_1(T_m) < g_2(T_m) \tag{105}$$

$$\dot{g}_1(T_1) > 0 \text{ for } T_m < T_1 < 0 \tag{106}$$

$$\dot{g}_2(T_1) < 0 \text{ for } T_m < T_1 < 0 \text{ if } e_o > 0 \text{ and } V_o > 0 \tag{107}$$

where a dot denotes differentiation with respect to T_1.

Proof

When $T_1 = T_m$, then $Y = 0$, $e_o = s_2$ and $e_1 = 1 - s_2$. From eq 101 through 103, we obtain

$$f_o = -s_2 Q / (1 - s_2) < 0 \tag{108}$$

$$f / T' = s_2\eta(1-\mu)Q / (e_1\alpha_o + s_2 s_3\mu Q) \geq 0. \tag{109}$$

It follows from eq 108 that $g_1 (T_m) < \sigma$.

Since $T_m < T_\sigma$ by eq 40, we will write eq 100 as

$$g_2(T_m) = \sigma + \int_{T_m}^{T_\sigma} \frac{K_2(s)}{K_1(s)} ds + \int_{T_m}^{0} \frac{f(s, T_1)}{K_1(s)T'(s, T_1)} ds. \tag{110}$$

Since the second and the third terms in the right side of eq 110 are positive, we find that $g_2 (T_m) > \sigma$; eq 105 holds true.

Differentiating eq 99 with respect to T_1, we obtain

$$\dot{g}_1(T_1) = (\delta_o / K_o)(\alpha_o / \eta)\left(e_3 / e_1^2\right)\dot{y} \tag{111}$$

where e_3 is defined as

$$e_3 = e_1 + s_2 s_3(Q - Y) / \alpha_o = \left[(1 - s_2)\alpha_o + s_2 s_3 Q\right] / \alpha_o > 0. \tag{112}$$

It follows from eq 111 that 106 holds true.

Differentiating eq 100 with respect to T_1, we obtain

$$\dot{g}_2(T_1) = -\left(\eta / \alpha_o^2\right)\left(e_3 / e_1^2\right)\int_{T_1}^{0} \frac{W(s, T_1)}{K_1(s)[(1 + e_4\mu(s, T_1)]^2} ds \tag{113}$$

where e_4 and W are defined as

$$e_4(T_1) = (s_2 s_3 / \alpha_o)(Q - Y) / e_1 \tag{114}$$

$$W(s, T_1) = \left[\alpha_o(1 - s_2\mu) + s_2 s_3\mu Q\right]\dot{Y} + e_o\alpha_o\dot{\mu}(Q - Y). \tag{115}$$

Since $V_o > 0$, by eq 92, $Q - Y > 0$. Differentiating eq 104 with respect to T_1, we obtain

$$\dot{\mu}(s, T) = [w_o - \nu(T_1)]^{-1}\mu\, \dot{\nu}(T_1) = \mu_o(T_1)\mu \geq 0. \tag{116}$$

Since $e_o > 0$, so $W(s,T_1) > 0$. Hence, eq 107 holds true. ❑

It is noted that $\dot{g}_1(T_1)$ and $\dot{g}_2(T_1)$ may be discontinuous at $T_1 = T_\sigma$ due to the singularity of $K_2 (T_1)$. We will consider a straight line L (α_{1s}, α_o) in S_p defined as

$$L(\alpha_{1s}, \alpha_o) = [(\alpha_1, \alpha_o): \alpha_1 \geq \alpha_{1s}] \tag{117}$$

where $(\alpha_{1s}, \alpha_o) \in C_s$. We will study the behavior of T_1 and V_o on $L(\alpha_{1s}, \alpha_o)$. Differentiating eq 92 with respect to Q, we obtain

$$e_1(w_o - \nu_1)\rho_{30}\tilde{V}_o = 1 - e_2\tilde{T}_1 \tag{118}$$

where a tilde denotes differentiation with respect to Q for a given α_o and e_2 is defined as

$$e_2(T_1) = (e_3 \,/\, e_1)\dot{Y} - \mu_o(Q - Y). \tag{119}$$

Differentiating eq 98 with respect to Q, we obtain

$$E_1(T_1)\tilde{T}_1 = E_2(T_1) \tag{120}$$

where E_1 and E_2 are defined as:

$$E_1(T_1) = \dot{g}_1(T_1) - \dot{g}_2(T_1) \tag{121}$$

$$E_2(T_1) = (e_o \,/\, e_1)\left[(\delta_o \,/\, K_o) + (\eta \,/\, \alpha_o)\int_{T_1}^{0} \frac{1 - \mu(s, T_1)}{K_1(s)[1 + e_4\mu(s, T_1)]^2} ds\right]. \tag{122}$$

Now we will begin our search of solutions of eq 92, 93 and 98 in S_p with a special case in which e_o vanishes. For the sake of brevity we will refer the problem of eq 92, 93 and 98 to as Problem P hereafter.

Proposition 5

There exists a unique solution of Problem P such that $e_o = 0$ and $V_o > 0$ if the following condition holds true:

$$\sigma < \int_{T_x}^{0} \frac{K_2(s) - (\eta \,/\, s_3)}{K_1(s)} ds. \tag{123}$$

Proof

When $e_o = 0$, then $T_1 = T_x$ and from eq 93 we obtain

$$f_o = Y = \alpha_o \,/\, s_3. \tag{124}$$

Using eq 101, 102 and 124, we will reduce eq 98 to

$$\alpha_o\delta_o \,/\, (s_3K_o) = \int_{T_x}^{0} \frac{K_2(s) - (\eta \,/\, s_3)}{K_1(s)} ds - \sigma. \tag{125}$$

Since eq 123 holds true, for a given σ there exists a unique and positive $\alpha_o(\sigma)$ that satisfies eq 125.

We will denote $\alpha_o(\sigma)$ by α_{oc} , and consider a line $L_c(\alpha_{1s},\alpha_{oc})$ defined as

$$L_c(\alpha_{1s}, \alpha_{oc}) = \{(\alpha_1, \alpha_{oc}) \;:\; \alpha_1 \geq \alpha_{1s}\}. \tag{126}$$

Our aim is to show $V_o > 0$ on L_c. From eq 122 we find that $E_2(T_1)$ vanishes if and only if $T_1 = T_x$ or $e_o = 0$. From eq 111 and 113 we find that $\dot{g}_1(T_x) > 0$ and $\dot{g}_2(T_x) < 0$; hence $E_1(T_x) > 0$. Therefore, $\tilde{T}_1$ vanishes in this case. Using eq 118, we find that $\tilde{V}_o$ is positive. Therefore, V_o is positive on L_c except at a point $(\alpha_{1s},\alpha_{oc})$ where V_o vanishes. ❑

We will define $g_o(\sigma)$ as

$$g_o(\sigma) = \int_{T_x}^{0} \frac{K_2(s) - (\eta \,/\, s_3)}{K_1(s)} ds - \sigma \tag{127}$$

or we may write g_o as

$$g_o(\sigma) = \int_{T_x}^{T_\sigma} \frac{K_2(s)}{K_1(s)} ds - (\eta / s_3) \int_{T_x}^{0} \frac{1}{K_1(s)} ds \tag{128}$$

where $T_\sigma = -\sigma/\gamma$ as defined by eq 37. We will define σ_x as

$$\sigma_x = -\gamma T_x \, . \tag{129}$$

It follows from eq 128 that $T_x < T_\sigma$ or $\sigma_x > \sigma$ when eq 123 holds true.

When $T_x < T_\sigma$, $g_o(\sigma)$ is a decreasing and continuous function of σ. We will define σ_c as

$$g_o(\sigma_c) = \int_{T_x}^{T_c} \frac{K_2(s)}{K_1(s)} ds - (\eta / s_3) \int_{T_x}^{0} \frac{1}{K_1(s)} ds = 0, \quad \text{or} \quad \sigma_c = \int_{T_x}^{0} \frac{K_2(s) - (\eta / s_3)}{K_1(s)} ds \tag{130}$$

where T_c is defined as

$$T_c = -\sigma_c/\gamma > T_x. \tag{131}$$

It is noted that T_c is uniquely determined by the properties of a given soil and that $T_c > T_x$ or $\sigma_c < \sigma_x$. Using g_o, when $\sigma < \sigma_c$, we will write α_{oc} of Proposition 5 as

$$\alpha_{oc} = (s_3 K_o \,/\, \delta_o)(\sigma_c - \sigma). \tag{132}$$

When $\sigma < \sigma_c$, the $L_c\,(\alpha_{1s}, \alpha_{oc})$ divides S_p into two regions $S_p^+(e_o > 0)$ and $S_p^-(e_o < 0)$ (Fig. 2b). The region S_p^- disappears if $\sigma \geq \sigma_c$. In terms of α_o we may define S_p^+ and S_p^- as

$$S_p^+ = \left\{ (\alpha_1, \alpha_o) \in S_p : \alpha_o > \alpha_{oc}, \; e_o > 0 \right\} \tag{133}$$

$$S_p^- = \left\{ (\alpha_1, \alpha_o) \in S_p : \alpha_o < \alpha_{oc}, \; e_o < 0 \right\}. \tag{134}$$

SOLUTIONS IN $\mathbf{S_p^+}$ WHEN $\sigma < \sigma_c$

We will consider a line L_c^+ defined as

$$L_c^+(\alpha_{1s}, \alpha_o) = \left\{ (\alpha_1, \alpha_o) : \alpha_1 > \alpha_{1s} \text{ and } \alpha_o > \alpha_{oc} > 0 \right\}. \tag{135}$$

It is clear that L_c^+ belongs to S_p^+ where $T_1 < T_x$.

Proposition 6

In S_p^+ $T_1 < T_x < T_\sigma$ if $\sigma < \sigma_c$.

Proof

Suppose that there exists a point in S_p^+ such that $T_x \geq T_\sigma$. We will write eq 128 as

$$\sigma_c = \sigma - \gamma(T_x - T_\sigma) - (\eta / s_3) \int_{T_x}^{0} \frac{1}{K_1(s)} ds. \tag{136}$$

It follows from eq 136 that $\sigma > \sigma_c$. This contradicts the assumption. ❑

Proposition 7

Suppose that a solution of Problem P exists on L_c^+, then $V_o > 0$ and $\tilde{T}_1 > 0$ on L_c^+.

Proof

First we will examine the behavior of V_o in a neighborhood of $\alpha_1 = \alpha_{1s}$ on L_c^+. When α_1 approaches α_{1s}, then $(Q - Y)$ approaches zero. Since e_3 approaches e_1, eq 118 is reduced to:

$$e_1(w_o - \nu_1)\rho_{10}\tilde{V}_o = 1 - \dot{Y}\tilde{T}_1. \tag{137}$$

Also eq 120 is reduced to

$$\dot{Y}\tilde{T}_1 E_3(T_1) = E_4(T_1) \tag{138}$$

where E_3 and E_4 are defined as

$$E_3(T_1) = 1 + (K_o / \delta_o)(\eta / \alpha_o)\int_{T_1}^{0} \frac{1 - e_o\mu}{K_1(1 + e_4\mu)^2} ds \tag{139}$$

$$E_4(T_1) = e_o\left[1 + (K_o / \delta_o)(\eta / \alpha_o)\int_{T_1}^{0} \frac{1 - \mu}{K_1(1 + e_4\mu)^2} ds\right]. \tag{140}$$

Since $0 < e_o < 1$, so $E_3 > E_4 > 0$ or $\dot{Y}\tilde{T}_1 < e_o < 1$. Hence we find that V_o is positive in a neighborhood of $\alpha_1 = \alpha_{1s}$. Suppose that there is a point on L_c^+ where $V_o < 0$. Since $V_o\,(Q)$ is continuous on L_c^+ for $T_1 < T_\sigma$, one can find a point on L_c^+ such that V_o vanishes. However, this contradicts Prop. 1. Therefore, V_o must be positive on L_c^+. Since $e_o > 0$ and $V_o > 0$ on L_c^+, in eq 120 $E_2\,(T_1) > 0$ and $E_1\,(T_1) > 0$ by eq 101. Therefore, $\tilde{T}_1 > 0$ on L_c^+ (α_{1s}, α_o) and on C_s. ❑

When $e_o > 0$, a solution with negative f_o may exist by eq 96. We will study such a possibility below.

Proposition 8

In S_p^+ f_o is positive if $\sigma < \sigma_c$.

Proof

Suppose that there exists a solution $T_1 = T_p$ of Problem P on L_c^+ (α_{1s}, α_o) such that $f_o\,(T_p) = 0$ or $Y\,(T_p) = e_o\,(T_p)Q$. From eq 101 and 102 we obtain

$$T'(s, T_p) = -(1/\eta)\left[\alpha_o + s_2 s_3 \mu(s, T_p) Q\right] \tag{141}$$

$$f(s, T_p) = s_2\mu(s, T_p)Q. \tag{142}$$

T_p is a solution of the following equation given as

$$\sigma = g_1\{T_1 : f_o = 0\} = g_3(T_1) \tag{143}$$

where g_3 is defined as

$$g_3(T_1) = \int_{T_1}^{0} \frac{k_2(s)}{k_1(s)} ds - (\eta / s_3)\int_{T_1}^{0} \frac{\hat{e}_4\mu(s, T_1)}{K_1(s)[1 + \hat{e}_4\mu(s, T_1)]} ds \tag{144}$$

where $\hat{e}_4$ is defined as

$$\hat{e}_4 = (s_2 s_3 / \alpha_o)\,Q = s_2 s_3 Y / (\alpha_o e_o). \tag{145}$$

It is easy to find

$$g_3(T_m) = \sigma + \int_{T_m}^{T_\sigma} \frac{K_2(s)}{K_1(s)}\, ds > \sigma. \tag{146}$$

Differentiating eq 144 with respect to T_1, we obtain

$$g_3(T_1) = -(\eta / s_3)\int_{T_1}^{0} \frac{\hat{e}_4\dot{\mu}}{K_1(s)(1+\hat{e}_4\mu)^2}\, ds. \tag{147}$$

It follows from eq 147 that $\dot{g}_3(T_1) < 0$. Since $g_3(T_1)$ is continuous and $T_m < T_1 < T_x < T_\sigma$, for a solution $T_1 = T_p$ of eq 143 to exist we must have

$$\sigma - g_3(T_x) > 0. \tag{148}$$

When T_1 approaches T_x, e_o approaches zero and $g_3\,(T_1)$ approaches σ_c. Hence eq 148 is reduced to

$$\sigma > \sigma_c. \tag{149}$$

It is clear that eq 149 does not hold in this case. Hence the solution T_p does not exist on L_c^+. Since $f_o > 0$ in a neighborhood of $\alpha_1 = \alpha_{1s}$ on L_c^+ and $f_o\,(Q)$ is continuous on L_c^+ for $T_1 < T_\sigma$, f_o is positive on L_c^+. Since α_o is an arbitrary positive number, f_o is positive in S_p^+ if $\sigma < \sigma_c$. ❑

Proposition 9

There exists a unique solution of Problem P in S_p^+.

Proof

We will consider a line L_c^+ defined by eq 135. In view of Proposition 4 we need to show that $g_1\,(T_x) > g_2\,(T_x)$. From eq 99 and 100 we obtain

$$g_1(T_x) = \alpha_o\delta_o \,/\, (s_3K_o) \tag{150}$$

$$g_2(T_x) = \sigma_c. \tag{151}$$

For the sake of convenience we will define $W_o\,(T)$ for $T < 0$ as

$$W_o\,(T) = g_1\,(T) - g_2\,(T). \tag{152}$$

From eq 150 and 151 we obtain

$$W_o(T_x) = \alpha_o\delta_o \,/\, (s_3K_o) + \sigma - \sigma_c. \tag{153}$$

Using eq 132, we will reduce eq 153 to

$$W_o(T_x) = \delta_o(\alpha_o - \alpha_{oc}) \,/\, (s_3K_o). \tag{154}$$

Since $\alpha_o > \alpha_{oc}$, we find that $g_1\,(T_x) > g_2\,(T_x)$. Since $g_1\,(T_1)$ and $g_2\,(T_1)$ are continuous for $T_1 < T_\sigma$, there exists a unique solution of Problem P on L_c^+. Since α_o is arbitrary, for any given point $(\alpha_1\ \alpha_o)$ in S_p^+ there exists a unique solution. ❑

When $V_o > 0$, unfrozen water may exist in R_2 $(T < T_1)$ and the amount of unfrozen water in R_2 depends on T. Hence, the rate of heave depends on T and is given for $T \le T_1$ as

$$d_2 r(T) = f_o + s_2[\rho_{10} - \rho_{30}\nu(T)]V_o. \tag{155}$$

Using eq 87, we will reduce eq 155 to

$$d_2 r(T) = Y + s_2\rho_{30}[\nu_1 - \nu(T) + (s_3/\eta)y\hat{\nu}_1]\,V_o \tag{156}$$

where $\hat{\nu}_1$ is defined as

$$\hat{\nu}_1\ (T_1) = w_o - \nu(T_1). \tag{157}$$

It follows from eq 156 that r is positive for $T > T_m$. In engineering practices the frost heave ratio h and the water intake ratio h_w are often used. These are defined as

$$h = r\,(T\,)\,/\,V_o \tag{158}$$

$$h_w = f_o\,/\,V_o. \tag{159}$$

According to M_1, h and h_w are given as

$$h = \alpha_o y\,/\,(d_2\eta V_o) + (s_2\rho_{30}\ /\ d_2)[\nu_1 - \nu(T) + (s_3\ /\ \eta)y\hat{\nu}_1] \tag{160}$$

$$h_w = \alpha_o y\,/\,(\eta V_o) - e_o\rho_{30}\hat{\nu}_1. \tag{161}$$

SOLUTIONS IN S_p^+ WHEN $\sigma \geq \sigma_c$

When $\sigma \geq \sigma_c$, the region S_p^- defined by eq 130 disappears, so $S_p = S_p^+$. We will consider a line L^+ (Fig. 2c) defined as

$$L^+(\alpha_{1s}, \alpha_o) = [(\alpha_1, \alpha_o)\!: \alpha_1 > \alpha_{1s} \text{ and } \alpha_o > 0] \tag{162}$$

where $(\alpha_{1s}, \alpha_o) \in C_s$.

Proposition 10

Suppose that there exists a solution T_1 of Problem P on L^+ such that $V_o > 0$. Then $T_1 < T_\sigma$ if $f_o \geq 0$, while T_1 may be greater than T_σ if $f_o < 0$.

Proof

We will write eq 98 as

$$\left[(\delta_o\ /\ K_o) - \int_{T_1}^{0} \frac{1}{K_1(s)T'(s,\,T_1)}\,ds\right] f_o.$$

$$= \int_{T_1}^{T_\sigma} \frac{K_2(s)}{K_1(s)}\,ds + (s_2\ /\ e_1)(Q - Y)\int_{T_1}^{0} \frac{\mu(s,T_1)}{K_1(s)T'(s,T_1)}\,ds. \tag{163}$$

Since $V_o > 0$, so $(Q - Y) > 0$. It follows from eq 163 that $T_1 < T_\sigma$ if $f_o \geq 0$, while T_1 may be greater than T_σ if $f_o < 0$. ❑

Since $e_o > 0$, $T_1 < T_x$. There are two cases: Case 1. $\sigma_c \leq \sigma < \sigma_x$ or $T_\sigma > T_x > T_1$ and Case 2.

$\sigma \geq \sigma_x$ or $T_\sigma \leq T_x$. First we will study Case 1. In Case 1 if a solution of Problem P exists on L^+, then $V_o > 0$ and $\tilde{T}_1 > 0$ on L^+ by the same reasoning as used in the proof of Proposition 7.

Proposition 11

There exists a unique solution $T_1 < T_x$ of Problem P on L^+ if $\sigma < \sigma_x$.

Proof

For a given point (α_1, α_o) on L^+, we will search a solution of Problem P under the assumption that $V_o > 0$ at this point. Because of eq 105, 106 and 101 for a unique solution T_1 to exist we must have

$$W_o(T_x) > 0. \tag{164}$$

Using eq 99 and 100 we obtain

$$W_o(T_x) = \alpha_o \delta_o / (s_3 K_o) + \sigma - \sigma_c > 0. \tag{165}$$

Hence, there exists a unique solution $T_1 < T_x$ of Problem P on L^+ if $V_o > 0$. But if such a solution exists, then $V_o > 0$. Therefore, there exists a unique solution of Problem P. ❑

Proposition 12

There exists a unique solution $T_1 = T_p$ of Problem P on L^+ such that $f_o(T_p) = 0$ if $\sigma < \sigma_x$.

Proof

It is easy to see that T_p is also a solution of eq 143 of Prop. 8. In this case $\sigma \geq \sigma_c$. Hence, there exists a unique solution T_p of eq 143. Since $f_o = 0$, $V_o > 0$ by eq 90. We will show that this solution is located on L^+. We will denote the point such that $T_1 = T_p$ by (α_{1p}, α_o) and the point on C_s by (α_{1s}, α_o). Since $f_o = 0$ or $Y - e_o Q = 0$ at (α_{1p}, α_o), we obtain

$$\alpha_{1p} = (1/k_1)\left[k_o + LK_2(T_p)/\left\{\eta e_o(T_p)\right\}\right]\alpha_o \tag{166}$$

where $e_o(T_p)$ is given as

$$e_o(T_p) = s_2\left[1 - (s_3/\eta)K_2(T_p)\right]. \tag{167}$$

From eq 69 we obtain

$$\alpha_{1s} = (1/k_1)[k_o + LK_2(T_s)/\eta]\alpha_o. \tag{168}$$

Using eq 166 and 168, we obtain

$$\alpha_{1p} - \alpha_{1s} = \left[K_2(T_p) - e_o(T_p)K_2(T_s)\right]L/\left[k_1 \eta e_o(T_p)\right]. \tag{169}$$

Since $\tilde{T}_1 > 0$ on L^+, $K_2(T_p) > K_2(T_s)$. Also $1 > e_o(T_p) > 0$. Hence, $\alpha_{1p} > \alpha_{1s}$. Therefore, the point (α_{1p}, α_o) is on L^+. ❑

The unique solution T_1 of Proposition 12 satisfies eq 143. We may write eq 143 and 145 as

$$\sigma = g_3(T_p) \tag{170}$$

$$\hat{e}_4(T_p) = s_2 s_3 K_2(T_p)/\left\{\eta e_o(T_p)\right\}. \tag{171}$$

It is easy to see from eq 144 that T_p depends on neither α_o nor α_1. Differentiating eq 170 with respect to σ, we find that T_p is a decreasing function of σ.

We will introduce a straight line L_p starting from the origin (Fig. 2c) defined as

$$L_p = \left\{(\alpha_1, \alpha_o): \alpha_o = k_1\left[k_o + LK_2(T_p) / \left\{\eta e_o(T_p)\right\}\right]^{-1} \alpha_1\right\}. \tag{172}$$

The line L_p divides the region S_p^+ into two regions S_x and S_{pp} defined as

$$S_x = \left\{(\alpha_1, \alpha_o): \alpha_o < k_1\left[k_o + LK_2(T_p) / \left\{\eta e_o(T_p)\right\}\right]^{-1} \alpha_1\right\} \tag{173}$$

$$S_{pp} = \left\{(\alpha_1, \alpha_o): \alpha_o > k_1\left[k_o + LK_2(T_p) / \left\{\eta e_o(T_p)\right\}\right]^{-1} \alpha_1\right\}. \tag{174}$$

It is clear that $f_o < 0$ in S_x while $f_o > 0$ in S_{pp}

Now we will study Case 2 where $T_\sigma \leq T_x$. In this case a solution T_1 may be greater than T_σ by Proposition 10. First we will search a solution T_1 of Problem P such that $T_1 < T_\sigma \leq T_x$ on L^+ under the assumption of $V_o > 0$. If such a solution exists, then $V_o > 0$ and $\tilde{T}_1 > 0$ on L^+ by the same reasoning as used in the proof of Proposition 7.

Proposition 13

There exists a unique solution T_1 of Problem P on L^+ such that $T_1 < T_\sigma \leq T_x$ and $f_o \geq 0$ if $\sigma \geq \sigma_x$.

Proof

First we assume that $V_o > 0$. Because of eq 105, 106, and 107 for a unique solution T_1 to exist we must have:

$$W_o(T_\sigma-) = g_1(T_\sigma-) - g_2(T_\sigma-) > 0. \tag{175}$$

From eq 99 and 100 we obtain:

$$W_o = a_o(T_\sigma-)f_o - (s_2 / e_1)(Q - Y)\int_{T_\sigma-}^{0} \frac{\mu(s, T_\sigma)}{K_1(s)T'(s, T_\sigma-)} ds. \tag{176}$$

where a_o is defined as

$$a_o(T) = (\delta_o / K_o) + (\eta / \alpha_o)\int_{T}^{0} \frac{1}{K_1(s)[1 + e_4\mu(s, T)]} ds \tag{177}$$

where f_o, e_1 and Y are functions of $T_\sigma-$, and e_4 defined by eq 114 is given as

$$e_4(T_\sigma-) = (s_2 s_3 / \alpha_o)[Q - Y(T_\sigma-)] / e_1(T_\sigma-). \tag{178}$$

From eq 177 we find that $a_o > 0$. Since $V_o > 0$, so $(Q - Y) > 0$. When $f_o \geq 0$, $W_o > 0$. Therefore, a unique solution T_1 of Problem P such that $f_o \geq 0$ exists on L^+ if $V_o > 0$.

We will denote one of such unique solution T_1 on L^+ such that $f_o = 0$ by T_p. Let T_p be located at (α_{1p}, α_o) on L^+. Since $f_o > 0$ in a neighborhood of C_s in S_p^+, the point (α_{1p}, α_o) must be in S_p^+. We will consider a segment $\ell_p(\alpha_o)$ of L^+ defined as

$$\ell_p(\alpha_o) = \left[(\alpha_1, \alpha_o): \alpha_{1s} < \alpha_1 < \alpha_{1p}\right]. \tag{179}$$

Since $f_o > 0$ on $\ell_p(\alpha_o)$, there exists a unique solution $T_1 < T_\sigma$ on $\ell_p(\alpha_o)$. Since $V_o > 0$ in a neighborhood of C_s in S_p^+ and $V_o(Q)$ is continuous on $\ell_p(\alpha_o)$, $V_o > 0$ on $\ell_p(\alpha_o)$. Therefore, there exists a unique solution T_1 of Problem P on $\ell_p(\alpha_o)$ such that $f_o > 0$. When $f_o = 0$, $V_o > 0$ because from eq 92 and 93 we find

$$Q - Y = e_1(Q - f_o) = e_1 Q > 0. \quad \square \tag{180}$$

We will define L_p by eq 172 where T_p is the unique solution in Prop. 13. Then the line L_p divides the region S_p^+ into two regions, S_x and S_{pp} defined by eq 173 and 174, respectively. Proposition 13 implies that there exists a unique solution in S_{pp} such that $f_o > 0$. We will search a solution T_1 of Problem P in S_x such that $T_1 < T_\sigma \le T_x$ and $f_o < 0$. It is easy to find from eq 180 that $V_o > 0$ when $f_o < 0$.

When $f_o < 0$, we will study the behavior of $W_o(T)$ for $T_p \le T \le T_x$. When $T = T_p$, $f_o = 0$. From eq 176 we obtain

$$W_o(T_p) = (s_2\eta / \alpha_o)Q \int_{T_p}^{0} \frac{\mu(s, T_p)}{K_1(s)[1 + e_4\mu(s, T_p)]} ds > 0. \tag{181}$$

Using eq 121, we obtain:

$$\dot{W}_o(T) = \dot{g}_1(T) - \dot{g}_2(T) = E_1(T). \tag{182}$$

Because of eq 106 and 107 we find

$$W_o(T) > 0 \quad \text{for} \quad T_p \le T \le T_x. \tag{183}$$

We may write eq 183 as

$$W_o(T_\sigma -) > 0 \quad \text{for} \quad T_p < T_\sigma \le T_x. \tag{184}$$

We will consider L^+ defined by eq 162. We have found that there exists a unique solution T_1 of Problem P for $\alpha_{1s} \le \alpha_1 \le \alpha_{1p}$. As α_1 increases from α_{1s} to α_{1p}, T_1 increases from T_s to T_p and f_o decreases from f_s to zero. Because of eq 184 there exists a unique solution T_1 of Prob. P for $\alpha_{1p} < \alpha_1$ such that $T_p < T_1 < T_\sigma \le T_x$ and $f_o < 0$. We will state our findings below.

Proposition 14

There exists a unique solution T_1 of Problem P on L^+ with $\alpha_{1p} < \alpha_1$ such that $T_p < T_1 < T_\sigma$ and $f_o < 0$ if $\sigma \ge \sigma_x$.

As stated above there may exist a solution T_1 of Problem P such that $T_1 > T_\sigma$ in Case 2. But we are not certain of the existence of such a solution.

SOLUTION IN S_p^-

We will consider a line L_c^- defined as

$$L_c^-(\alpha_{1s}, \alpha_o) = \{(\alpha_1, \alpha_o) : \alpha_1 > \alpha_{1s} \quad \text{and} \quad \alpha_{oc} > \alpha_o > 0\} \tag{185}$$

where $(\alpha_{1s}, \alpha_o) \in C_s$ and (α_1, α_{oc}) is on the line L_c (Fig. 2b). It is clear that L_c^- belongs to S_p^- where $e_o < 0$.

Proposition 15

There exists at least one solution T_1 of Problem P on L_c^- such that $T_x < T_1 < T_s < T_\sigma$, $V_o > 0$ and $f_o > 0$.

Proof

Since $e_o < 0$ in this case, we must search a solution T_1 on L_c^- such that $T_1 > T_x$. First we examine eq 98 when $T_1 = T_x$. Since $\alpha_o < \alpha_{oc}$ in this case, using eq 154, we find that $g_1(T_x) < g_2(T_x)$.

We will assume that $V_o > 0$ and $f_o > 0$ for the time being. From eq 99 and 100 we obtain:

$$W_o(T_s) = a_o(T_s)f_o - a_1(T_s) + (\eta / \alpha_o)a_2(T_s) \tag{186}$$

where a_o is defined by eq 177, and a_1 and a_2 are defined as

$$a_1(T) = \int_T^{T_\sigma} \frac{K_2(s)}{K_1(s)}\, ds \tag{187}$$

$$a_2(T) = (s_2 / e_1)(Q - Y)\int_T^0 \frac{\mu(s,T)}{K_1(s)[1 + e_4\mu(s,T)]}\, ds. \tag{188}$$

Since T_s is the solution of eq 98 when $V_o = 0$ ($Q = Y$), we obtain

$$W_o(T_s) = \left[(\delta_o / K_o + (\eta / \alpha_o)\int_{T_s}^0 \frac{ds}{K_1(s)}\right] f_s - a_1(T_s) = 0 \quad \text{if } V_o = 0 \tag{189}$$

where f_s is given by eq 68. Using eq 189, we will reduce eq 186 to

$$W_o(T_s) = (-1 / e_1)(\delta_o / K_o)E_4(T_s)(Q - Y) \tag{190}$$

where E_4 is defined by eq 140. It follows from eq 190 that $g_1(T_s) > g_2(T_s)$. Since g_1 and g_2 are continuous functions of T_1 for $T_1 < T_\sigma$, there exists at least one solution T_1 of Prob. P on L_c^- such that $T_x < T_1 < T_s < T_\sigma$ if $V_o > 0$ and $f_o > 0$.

When such a solution exists, from eq 137 and 138 we find that $\tilde{T}_1 < 0$ and $V_o > 0$ in a neighborhood of $\alpha_1 = \alpha_{1s}$ in S_p^-. By the similar reasoning to that used in the proof of Proposition 7, we find that $V_o > 0$ on L_c^-. Since $V_o > 0$ and $T_1 < T_\sigma$ on L_c^-, hence by Proposition 10 $f_o > 0$. ❑

It is clear that a solution of Proposition 15 is unique if $\dot{g}_2(T_1) \le 0$ on L_c^-. We will study the sign of $\dot{g}_2(T_1)$ below. We will write eq 113 as

$$\dot{g}_2(T_1) = -(1 / \alpha_o)(e_3 / e_1^2)W_1(T_1) \tag{191}$$

where W_1 is defined as

$$W_1(T_1) = (\eta / \alpha_o)\int_{T_1}^0 \frac{W(s,T_1)}{K_1(s)(1 + e_4\mu)^2}\, ds. \tag{192}$$

The sign of $\dot{g}_2(T_1)$ depends on that of $W_1(T_1)$. We will write eq 192 as

$$W_1(T_1) = w_1\alpha_o + w_2\alpha_1. \tag{193}$$

w_1 and w_2 are defined as

$$w_1(T_1) = \dot{y}a_3 + (e_1e_5 - \dot{y})a_4 > 0 \tag{194}$$

$$w_2(T_1) = s_1(\dot{y} - e_5)a_4 = [s_1\eta / (s_2 s_3 \hat{\nu}_1)]e_6 a_4 \tag{195}$$

$$s_1 = (k_1 / k_o)(1 - s_2) > 0 \tag{196}$$

$$e_5(T_1) = -\eta e_o \mu_o / (s_2 s_3) > 0 \tag{197}$$

$$e_6(T_1) = \frac{\partial}{\partial T_1}(-e_o \hat{\nu}_1) \tag{198}$$

$$a_3(T_1) = \int_{T_1}^{0} \frac{1}{K_1(s)(1+e_4\mu)^2} ds \tag{199}$$

$$a_4(T_1) = \int_{T_1}^{0} \frac{\mu}{K_1(s)(1+e_4\mu)^2} ds. \tag{200}$$

The sign of w_2 is the same as e_6 and e_6 is a property of a given soil. First we will consider the case where the following condition holds true:

$$e_6(T_1) \geq 0 \quad \text{for} \quad T_1 < T_s. \tag{201}$$

Suppose that T_1 is a solution of Proposition 15. Then $W_1(T_1) > 0$ and $\dot{g}_2(T_1) < 0$ on L_c^- in this case. Therefore, this solution is unique. From eq 120 we find that $T_1 < 0$ on L_c^-.

Next we will study the case in which eq 201 does not hold true. We will examine the behavior of $W_1(T_1)$ on L_c^-. When α_1 approaches α_{1s}, T_1 approaches T_s, α_{1s} is given by eq 168. Using eq 88, we reduce eq 168 to

$$\alpha_{1s} = (e_{1s}/s_1)\alpha_o \tag{202}$$

where

$$e_{1s} = e_1(T_s). \tag{203}$$

We will write W_1 as

$$W_1(T_1) = (a_3 - e_o a_4)\dot{y}\alpha_o + (s_1\alpha_1 - e_1\alpha_o)(\dot{y} - e_5)a_4 \tag{204}$$

when α_1 approaches α_{1s}, the second term of the right-hand side of eq 204 vanishes and $W_1(T_1)$ approaches $W_1(T_s)$ given as

$$W_1(T_s) = (a_3 - e_o a_4)\dot{y}\alpha_o > 0. \tag{205}$$

It follows from eq 205 that $W_1(T_1) > 0$ in a neighborhood of $\alpha_1 = \alpha_{1s}$ on L_c^-. It is easy to find that W_1 vanishes when α_1 becomes infinite.

Since the second term of the right-hand side of eq 204 is negative on L_c^-, it is possible that $W_1(T_1)$ may become negative at some point on L_c^-. If such is the case, then there exists at least one point (α_{1g}, α_o) on L_c^- where W_1 vanishes because W_1 is a continuous function of α_1. From eq 193 we obtain

$$\alpha_{1g} = -(w_1 / w_2)\alpha_o. \tag{206}$$

From eq 202 and 206 we obtain

$$(\alpha_{1g} - \alpha_{1s}) / \alpha_o = a_3\eta e_7[1 - (a_4 / a_3)e_{os}] / (-s_2 s_3 \hat{\nu}_1 w_2) \tag{207}$$

where e_7 is defined as

$$e_7(T_1) = e_6 - e_o \hat{v}_1 [1 - (a_4 / a_3) e_o] / [1 - (a_4 / a_3) e_{os}] \tag{208}$$

$$e_{os} = e_o(T_s). \tag{209}$$

It follows from eq 207 that $\alpha_{1g} > \alpha_{1s}$ if e_7 is positive. This implies that there exists a point (α_{1g}, α_o) on L_c^- such that W_1 vanishes and that W_1 may be negative for $\alpha_1 > \alpha_{1g}$. When W_1 is negative, then $\dot{g}_2(T_1)$ is positive and the uniqueness of the solution is not warranted. Hence, if e_7 is positive, the solution of Proposition 15 is unique under condition given as

$$e_6(T_1) < 0 \quad \text{and} \quad e_7(T_1) > 0 \quad \text{for} \quad T_1 < T_s \quad \text{and} \quad \alpha_1 \le \alpha_{1g}. \tag{210}$$

On the other hand, if $e_7 \le 0$, then $\alpha_{1g} \le \alpha_{1s}$. This clearly implies that $W_1 > 0$ on L_c^- and that a solution of Proposition 15 is unique. We will present our finding by the following proposition.

Proposition 16

The solution of Prop. 15 is unique if either $e_6(T_1) \ge 0$ or $e_7(T_1) \le 0$ for $T_1 < T_s$. When $e_7(T_1) > 0$ for $T_1 < T_s$, the solution is unique if eq 210 holds true.

APPLICATIONS

We will describe the use of traveling wave solutions obtained above for the empirical verification of the model M_1 below. It is known (Andersland and Anderson 1978) that the empirically determined function $\nu(T)$ under equilibrium conditions takes a form given as:

$$\nu(T) = A_o |T|^{-A_1} \qquad \text{for} \qquad T < 0 \tag{211}$$

where A_o and A_1 are positive constants. Experimental methods were proposed to determine K_1 (Williams and Burt 1974, Horiguchi and Miller 1983) and K_2 (Perfect and Williams 1980). Horiguchi and Miller (1983) empirically found that K_1 of several frozen porous media also takes the same form as eq 211. Since ν and K_1 are known to be bounded, we will use forms given as

$$\nu(T) = \begin{cases} w_o & A \le T < 0 \\ w_o (A / T)^{b_3} & A > T \end{cases} \qquad \text{if } V_o > 0 \tag{212}$$

$$K_1(T) = \begin{cases} K_o & A \le T < 0 \\ K_o (A / T)^{b_1} & A > T \end{cases} \tag{213}$$

where A is a small negative number, b_1 and b_3 are positive numbers. When $V_o = 0$, $\nu(T)$ is not needed in the balance equations of mass and heat. However, K_i ($i = 1,2$) implicitly takes account of the composition of the frozen fringe.

Recently, Nakano and Takeda (1994) empirically found that $K_2(T)$ of Kanto loam can be described in the same form as eq 211 for $T < T_\sigma$. Using eq 37, we will describe K_2 as

$$K_2(T) = \begin{cases} K_{20} = \gamma K_o & A \le T < 0 \\ \gamma\ K_1(T) & T_\sigma \le T < A \\ K_{20} (A / T)^{b_2} & T < T_\sigma. \end{cases} \tag{214}$$

The empirically determined values of parameters in eq 212, 213, and 214 for Kanto loam (Nakano and Takeda 1994) are $w_o = 0.740$, $A = -1.5 \times 10^{-4}$ °C, $b_3 = 0.110$, $K_o = 1.77 \times 10^3$ g / (cm·d·MPa), $b_1 = 0.520$, $K_{20} = 1.98 \times 10^3$ g / (cm·d·°C) and $b_2 = 1.04$.

The existence of the boundary C_s (Fig. 2a) has been verified empirically for three types of soils including Kanto loam under $\sigma = 0$ (Takeda and Nakano 1990). According to Prop. 2, T_s is a decreasing function of σ. This implies that the region S_i decreases as σ increases, which is also verified by the data of Kanto loam (Takeda and Nakano 1993). We will consider a freezing test in which a soil sample with a uniform initial temperature $T_a > 0$°C is frozen from the bottom up while the bottom temperature is kept constant at $T_b < 0$°C and the top temperature at T_a. The temperature field changes rapidly at the start of the test. However, as time elapses, the rate of the change slows down so that the transient freezing may be approximated by a series of quasi-steady freezing steps. Hence, the later part of the experiment can be represented by a trajectory in Figure 3, consisting of points $\alpha(t) = \{\alpha_1(t), \alpha_o(t)\}$ for $t_2 \geq t \geq t_o$.

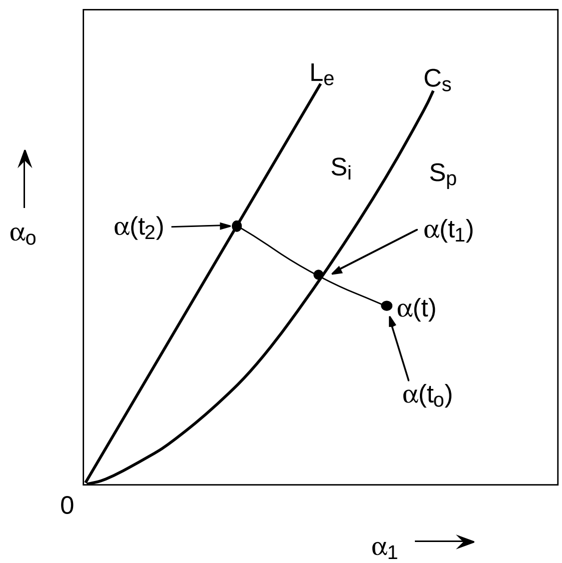


Figure 3. Trajectory of α(t).

A point α (t_o) is in S_p where frozen soil without any visible ice layer grows. As α_1 decreases and α_o increases with time, the trajectory approaches the vicinity of $\alpha(t_1)$ in S_p. The pattern of ice-rich frozen soil grown in this vicinity evidently depends on the soil type and the magnitude of α_1 (or α_o) (Takeda and Nakano 1990). The results of tests on Kanto loam, for instance, clearly indicate that the pattern of rhythmic ice banding is formed at the small values of α_1, while soil particles or small aggregates of soil particles are evenly dispersed at the greater values of α_1. The mechanism of such pattern formation is not well understood, but possible causes include the heterogeneity of soils and instabilities in the coupled heat and mass transport process with phase change. The quasi-steady solution described above predicts the average ice content of frozen soil and accounts for the growth of an ice layer but not the time-dependent formation of patterns.

When $\alpha(t)$ reaches $\alpha(t_1)$ on C_s, an ice layer emerges. While $\alpha(t)$ moves toward $\alpha(t_2)$ on L_e, the growth of the ice layer continues with decreasing growth rate until $\alpha(t)$ reaches $\alpha(t_2)$ on L_e where the ice layer stops growing. At $\alpha(t_2)$ f_o vanishes and we find from eq 59 and 60 that eq 76 is reduced to

$$\sigma = -\gamma \, T_\sigma. \tag{215}$$

This equation is often called the generalized Clausius-Clapeyron equation that describes the equality of chemical potentials of ice and water subjected to two different pressures. Radd and Oertle (1973) empirically validated eq 215.

The soil water is known to be expelled from the frost front under certain conditions depending upon soil type, stress level, etc., when a frost front advances through a saturated soil. Such a phenomenon is often called the pore water expulsion, and a concise review of papers on the subject was written by McRoberts and Morgenstern (1975). We have found that the unique solution of Problem P exists such that $f_o < 0$ if $\sigma \geq \sigma_c$. It is clear that the pore water expulsion occurs in this solution. We will examine the accuracy of M_1 by using experimental data of Kanto loam on water expulsion below.

Takashi et al. (1978) conducted numerous freezing tests similar to the test described above on overconsolidated samples of silt and clay to determine empirical descriptions for the heave ratio h and the water intake ratio h_w. In their tests, $T_a > 0°C$ was kept constant at a value 0.2–0.3°C higher than the freezing point of the sample, while T_b (t) was decreased with time from the initial value $T_b(0) = T_a$ in such a manner that V_o was kept nearly constant. After each test h and h_w were determined by measured total amounts of heave and water intake, respectively, for a given set of σ and V_o. The empirical descriptions obtained are given as

$$h = (m_1 / \sigma)[1 + (m_2 / V_o)^{1/2}] + m_o \tag{216}$$

$$h_w = d_2(m_1 / \sigma)[1 + (m_2 / V_o)^{1/2}] - s_2 m_3 \tag{217}$$

where m_i (i = 0,1,…, 3) are positive numbers that depend on a given soil. The sets of constants m_i for a few kinds of soils have been reported (Ohrai and Yamamoto 1991). Ryokai (1985) determined the set of constants m_i for Kanto loam by a series of freezing tests similar to those of Takashi et al. (1978). The values of m_i are $m_o = 0.0002$, $m_1 = 0.980$ kPa, $m_2 = 8.07 \times 10^3$ cm / d and $m_3 = 0.439$. In his tests (Ryokai 1985) the height of samples was 2.0 cm and T_a = 0.5°C.

In the Takashi's freezing test, σ and V_o are constants but α_o and δ_o vary with time. For instance, in the test by Ryokai (1985), the value of δ_o was 2.0 cm at the start and decreased with time. The value of α_o was 0.25°C / cm at the start, increased to about 1.0°C / cm when a quarter of a sample remained unfrozen, and increased further with time. It has been recognized (Ohrai and Yamamoto 1991) that the empirical formulas (eq 216 and 217) must be applied for cases where V_o is greater than about 1.5 cm / d, because the behavior of these formulas as V_o approaches zero is incompatible with empirical findings. It follows from eq 216 and 217 that r and f_o vanish as V_o vanishes. But in reality when V_o vanishes, an ice layer begins growing so that r and f_o do not vanish. It is also important to mention that the empirical formulas must be applied for cases where σ is greater than about 50 kPa.

Ideally, the results of Takashi's tests should be compared with the theoretical predictions based on the solution under the same initial and boundary conditions as those of actual tests. Since such unsteady solutions are not yet known, we will use eq 160 and 161 based on the traveling wave solutions studied above. Differentiating eq 98 with respect to σ, we obtain

$$E_1(T_1)\frac{\partial T_1}{\partial \sigma} = -1. \tag{218}$$

Since E_1 is positive, T_1 and y are decreasing functions of σ. It follows from eq 160 and 216 that h is a positive and decreasing function of σ and V_o. From eq 161 and 217 we find that h_w is also a decreasing function of σ and V_o and that h_w becomes negative when σ and V_o become large, namely, water expulsion occurs.

Assuming that $\eta = 1$, we will calculate several important parameters of Kanto loam as

$$s_3 = 4.25 \text{ cm} \cdot \text{d} \cdot {}^\circ\text{C/g}, \quad \sigma_c = 453 \text{ kPa}, \qquad \sigma_x = 998 \text{ kPa}. \tag{219}$$

We anticipate that water expulsion occurs if $\sigma > 453$ kPa. In order to calculate h and h_w by eq 160 and 161, respectively, we must determine T_1. Using eq 212, 213 and 214, we will reduce eq 98 to

$$F\{y(T_1), \alpha_o, V_o, \sigma, \delta_o\} = 0. \tag{220}$$

A detailed description of F is given elsewhere (Nakano and Primicerio 1995). Since eq 220 is a nonlinear algebraic equation, for given α_o, δ_o, σ and V_o, T_1 was calculated numerically by the Newton-Raphson method.

Calculating h and h_w as functions of V_o for various sets of (α_o, δ_o, σ) with the ranges of $0.1 \le \alpha_o \le 1.0$°C/cm, $0.5 \le \delta_o \le 5.0$ cm, and $0 \le \sigma \le 1.5$ MPa, we have found that the dependence of h and h_w on σ is the strongest, and then less strong on V_o, α_o and δ_o in order of decreasing dependence. The value of δ_o is proportional to the resistance against the flow of water in R_o. On the other hand, the flow resistance of R_1 increases with increasing σ. When the resistance of R_1 becomes much greater than that of R_o, the effect of δ_o diminishes. The effect of δ_o was found negligible when σ is greater than 300 kPa. The calculated values of h vs. V_o and h_w vs. V_o for Kanto loam under the condition of σ = 500 kPa, and δ_o = 1.0 cm with four different values of α_o are presented in Figure 4 and 5, where circles are values calculated by the empirical formulas for Kanto loam determined by Ryokai (1985). From Figure 4 and 5 we find that the effect of α_o on h and h_w is

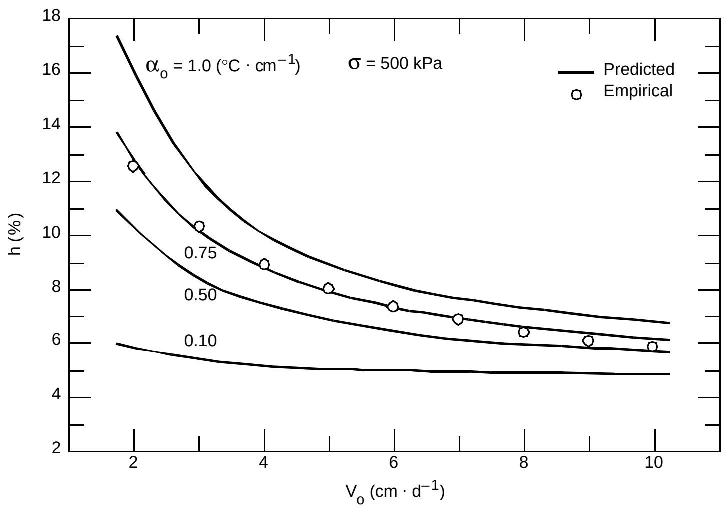


Figure 4. Calculated values of h (%) vs. V_o (cm/d) under four different values of α_o with δ_o = 1.0 cm and σ = 500 kPa. Circles are calculated by an empirical formula (Ryokai 1985).

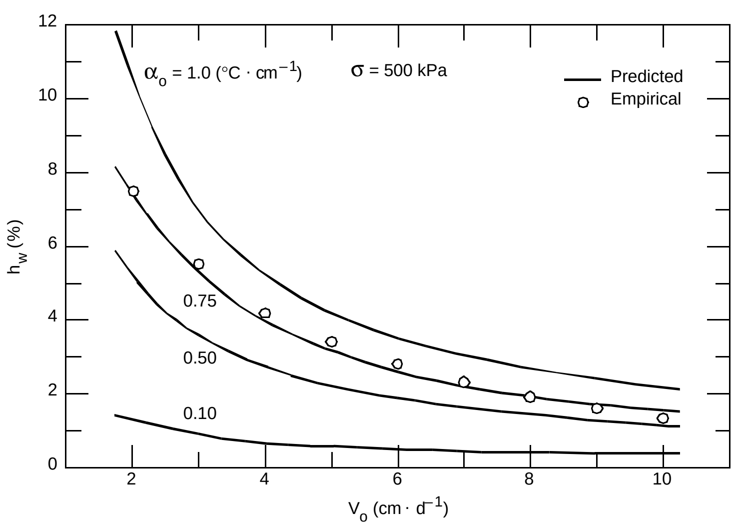


Figure 5. Calculated values of h_w (%) vs. V_o (cm/d) under four different values of α_o with δ_o = 1.0 cm and σ = 500 kPa. Circles are calculated by an empirical formula (Ryokai 1985).

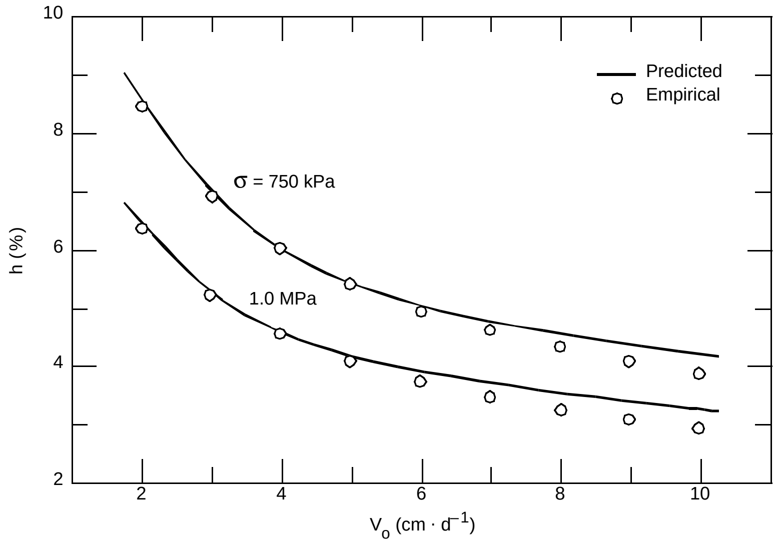


Figure 6. Calculated values of h (%) *vs.* V_0 *(cm/d) with* α_0 *= 0.75°C/cm,* δ_0 *= 1.0 cm, σ = 0.75, and 1.0 MPa. Circles are calculated by an empirical formula (Ryokai 1985).*

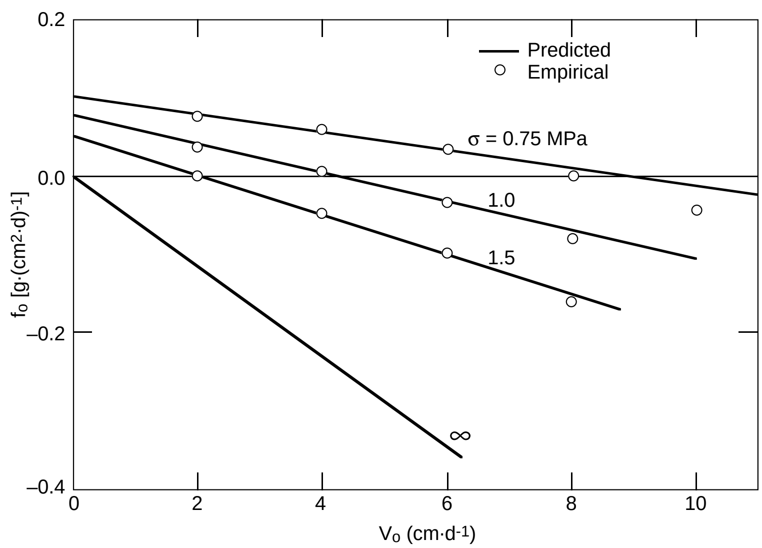


Figure 7. Calculated values of f_0 *[g/(cm^2 · d)] vs.* V_0 *(cm/d) with* α_0 *= 0.75°C/cm and* δ_0 *= 1.0 cm and σ = 0.75, 1.0, 1.5 MPa, and ∞. Circles are calculated by an empirical formula (Ryokai 1985).*

significant and that the calculated curves with α_o = 0.75°C / cm agree well with the empirical formulas.

Assuming that α_o = 0.75°C / cm and δ_o = 1.0 cm, we calculated y, h and h_w as functions of V_o with σ = 0.75 and 1.0 MPa. In Figure 6 predicted curves of h vs. V_o with σ = 0.75 and 1.0 MPa are presented. From this figure we find that the predicted curves of h vs. V_o tend to deviate from the empirical formulas when V_o > 6.0 cm / d. The calculated values of f_o vs. V_o are presented in Figure 7 when α_o = 0.75°C / cm, δ_o = 1.0 cm, and σ = 0.75, 1.0, 1.5 MPa and ∞. Circles in the figure are values calculated by the empirical formula. The values of V_o at f_o = 0 are 8.94 (7.77) cm / d, 4.19 (4.31), and 2.01 (1.88) for σ = 0.75, 1.0 and 1.5 MPa, respectively, where numbers in parentheses are calculated by the empirical formula. The agreement between the predicted and empirical values of V_o at f_o = 0 is satisfactory.

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13. ABSTRACT (*Maximum 200 words*)

The scientific study of soil freezing began in the early 1900s and an accurate mathematical description of the freezing process has been sought for nearly 80 years. Despite numerous publications on the subject, there is as yet no clear consensus on the mathematical model of soil freezing. In this report a mathematical model called $M_1$is presented. The existence of traveling wave solutions to the problem is shown. For a given fine-grained soil, such solutions are shown to exhibit three distinct behaviors depending on given thermal and hydraulic conditions. When a frost front (0°C isotherm) advances, water is either attracted to the front or expelled from it. Under certain conditions an ice layer containing hardly any soil particles grows. The report describes how the traveling wave solutions have been used for the empirical verification of M_1.

14. SUBJECT TERMS

Frost heave
Ice segregation
Soil freezing
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